# The $\beta$-transformation with a hole at $0$

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# Urbański's second paper (of 231 & counting)

*Ergod. Th. & Dynam. Sys.* (1986), **6**, 295–309
*Printed in Great Britain*

## On Hausdorff dimension of invariant sets for expanding maps of a circle

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(*Received* 5 *November* 1984 *and revised* 1 *July* 1985 *and* 10 *October* 1985)

*Abstract.* Given an orientation preserving $C^2$ expanding mapping $g: S^1 \to S^1$ of a circle we consider the family of closed invariant sets $K_g(\varepsilon)$ defined as those points whose forward trajectory avoids the interval $(0, \varepsilon)$. We prove that topological entropy of $g|K_g(\varepsilon)$ is a Cantor function of $\varepsilon$. If we consider the map $g(z) = z^q$ then the Hausdorff dimension of the corresponding Cantor set around a parameter $\varepsilon$ in the space of parameters is equal to the Hausdorff dimension of $K_g(\varepsilon)$. In § 3 we establish some relationships between the mappings $g|K_g(\varepsilon)$ and the theory of $\beta$-transformations, and in the last section we consider DE-bifurcations related to the sets $K_g(\varepsilon)$.

# Urbański's third paper

*Ergod Th & Dynam Sys* (1987), 7, 627–645
*Printed in Great Britain*

# Invariant subsets of expanding mappings of the circle

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(*Received* 7 *July* 1986 *and revised* 6 *January* 1987)

*Abstract* The continuity of Hausdorff dimension of closed invariant subsets $K$ of a $C^2$-expanding mapping $g$ of the circle is investigated If $g|K$ satisfies the specification property then the equilibrium states of Holder continuous functions are studied It is proved that if $f$ is a piecewise monotone continuous mapping of a compact interval and $\phi$ a continuous function with $P(f, \phi) > \sup(\phi)$, then the pressure $P(f, \phi)$ is attained on one-dimensional 'Smale's horseshoes', and some results of Misiurewicz and Szlenk [**M–Sz**] are extended to the case of pressure

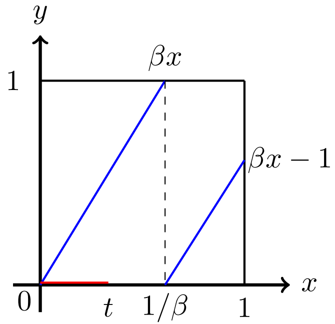
# Survivor set

Given $\beta \in (1,2]$, let

$$T_\beta : [0,1) \to [0,1); \quad x \mapsto \beta x \pmod 1.$$

For $t \in [0,1)$ we define the survivor set by

$$K_\beta(t) := \left\{ x \in [0,1) : T_\beta^n(x) \notin (0,t) \text{ for all } n \geq 0 \right\}.$$



## Some background

Urbański (1986, 1987) considered the case $\beta = 2$ and showed (among many other things – see later) that $\eta_2 : t \mapsto \dim_H K_2(t)$ is a non-increasing devil's staircase, with

$$\inf\{t > 0 : \eta_2(t) = 0\} = \frac{1}{2}. \quad (1)$$

Proof of (1).

- If $t > 1/2$, we take out the entire first branch of $T_2$, resulting in $K_2(t) = \{0\}$.
- If $t < 1/2$, the binary expansion of $t$ begins with $01^k0$ for some $k \in \mathbb{N}$. Then $K_2(t)$ contains all points whose binary expansion is an arbitrary concatenation of the words $01^{k+1}$ and $01^{k+2}$. This set has positive Hausdorff dimension.

$\square$

## Some related results

- Glendinning & Sidorov (2015) considered more general holes and showed that

$$K_2(a,b) := \{x \in [0,1) : T_2^n(x) \notin (a,b)\ \forall n \geq 0\}$$

  has positive Hausdorff dimension if $b - a < 0.175092$.
- Extended by Clark (2016) to $K_\beta(a,b)$ for $\beta \in (1,2)$.

## Greedy and quasi-greedy expansions

### Definition

For $\beta \in (1, 2]$ and $t \in (0, 1)$, let $b(t, \beta) \in \{0, 1\}^{\mathbb{N}}$ denote the greedy $\beta$-expansion of $t$ (i.e. the lexicographically largest expansion).

### Definition

For $\beta \in (1, 2]$, let $\alpha(\beta)$ denote the quasi-greedy $\beta$-expansion of 1 (i.e. the lexicographically largest expansion not ending in $0^\infty$).

### Example

Let $\beta = (1 + \sqrt{5})/2$. Then

$$1 = \frac{1}{\beta} + \frac{1}{\beta^2} = \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^5} + \dots,$$

so

$$b(1, \beta) = 110^\infty, \qquad \alpha(\beta) = (10)^\infty.$$

## The symbolic survivor set

### Theorem (Parry, 1960)

*A sequence* $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$ *is the greedy expansion of some point* $x$ *in base* $\beta$ *iff* $\sigma^n(\mathbf{x}) \prec \alpha(\beta)$ *for all* $n \geq 0$*. Here* $\sigma$ *is the shift map.*

### Theorem (Kalle, Kong, Langeveld & Li, 2020)

*Let*

$$\mathbf{K}_\beta(t) := \left\{ \mathbf{x} \in \{0,1\}^{\mathbb{N}} : b(t,\beta) \preccurlyeq \sigma^n(\mathbf{x}) \preccurlyeq \alpha(\beta) \quad \forall n \geq 0 \right\}.$$

*Then*

$$\dim_H K_\beta(t) = \frac{h_{top}(\mathbf{K}_\beta(t))}{\log \beta},$$

*where* $h_{top}(X)$ *denotes the topological entropy of a subshift* $X$*. Furthermore, the dimension function* $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ *is a non-increasing devil's staircase.*

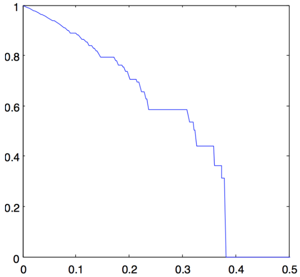



Figure: The graph of $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ with $\beta = \frac{1+\sqrt{5}}{2}$

It is of interest to determine the critical value

$$\tau(\beta) := \min\{t \in [0,1) : \dim_H K_\beta(t) = 0\}.$$

Example: Let $\beta = (1+\sqrt{5})/2$. Note that

$$b\left(1-\frac{1}{\beta},\beta\right) = b\left(\frac{1}{\beta^2},\beta\right) = 010^\infty,$$

so

$$\begin{aligned}\mathbf{K}_\beta\left(1-\frac{1}{\beta}\right) &= \{\mathbf{x} : 010^\infty \preccurlyeq \sigma^n(\mathbf{x}) \preccurlyeq (10)^\infty \ \forall n\} \\ &= \{(01)^\infty, (10)^\infty\}.\end{aligned}$$

Indeed, we can show that $\tau(\beta) = 1 - \frac{1}{\beta} \approx 0.382$.

### Proposition (Kalle, Kong, Langeveld & Li, 2020)

*We have*

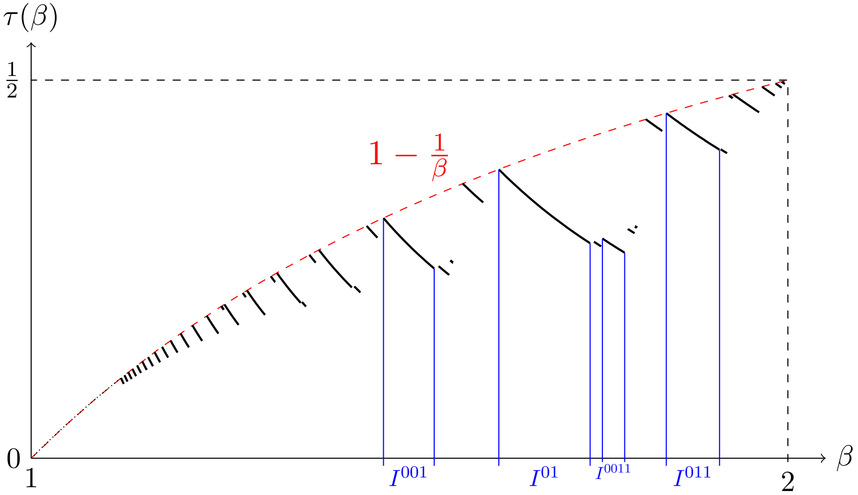
$$\tau(\beta) \le 1 - \frac{1}{\beta} \qquad \forall\, \beta \in (1,2],$$

*with equality for $\beta$ in an uncountable set of Hausdorff dimension $0$ (which we'll describe later).*

# Question

Question: Can we determine $\tau(\beta)$ for all $\beta \in (1, 2]$?

Answer: Yes, so don't fall asleep! But we'll need some more concepts.




The graph of the critical value function $\tau(\beta)$ for $\beta \in (1, 2]$.

# Farey words

To describe the critical values of $\tau(\beta)$ we need the Farey words:

$$F_0 = (0, 1),$$
$$F_1 = (0, 01, 1),$$
$$F_2 = (0, 001, 01, 011, 1),$$
$$F_3 = (0, 0001, 001, 00101, 01, 01011, 011, 0111, 1),$$
$$\cdots$$

Let

$$F^* := \bigcup_{n=1}^{\infty} F_n \setminus F_0.$$

Then each Farey word in $F^*$ begins with $0$ and ends with $1$.

Remark: There's a natural bijection between the Farey words and $\mathbb{Q} \cap [0, 1]$, e.g. $001 \leftrightarrow \frac{1}{3}$, $011 \leftrightarrow \frac{2}{3}$, etc.

# Lyndon words

## Definition

A finite word is Lyndon if it is aperiodic and it is the lexicographically smallest among its cyclical permutations.

Example: $001011$ is Lyndon, $0101$ and $01101$ are not.

Let $L^*$ be the set of Lyndon words of length $\geq 2$.

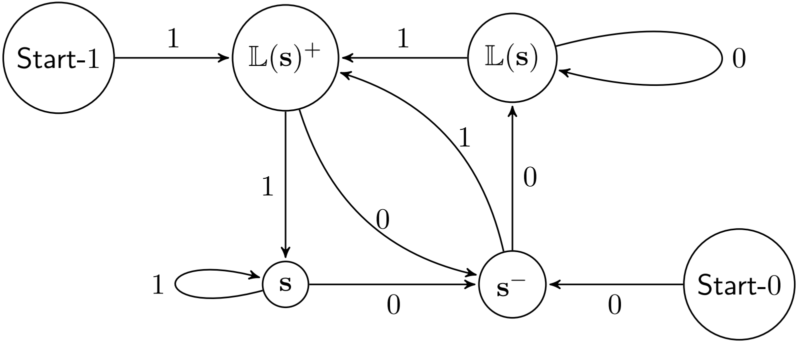
## Remark

- Any word in $L^*$ begins with $0$ and ends with $1$.
- $F^* \subsetneq L^*$. For example, $\mathbf{s} = 00111 \in L^* \setminus F^*$.

For a word $\mathbf{w} = w_1 \dots w_m \in \{0,1\}^*$, let $\mathbb{L}(\mathbf{w})$ be the lexicographically largest among all cyclical permutations of $\mathbf{w}$.

Example: $\mathbf{w} = 010110111 \quad \Rightarrow \quad \mathbb{L}(\mathbf{w}) = 111010110.$

# The substitution $\mathbf{s} \bullet \mathbf{r}$




Example. Let $\mathbf{s} = 01$. Then

$$\begin{aligned}
\mathbf{s} \bullet 01011 &= \mathbf{s}^-\mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})^+\mathbf{s} &&= 00\,11\,00\,11\,01,\\
\mathbf{s} \bullet 10010 &= \mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})\mathbb{L}(\mathbf{s})^+\mathbf{s}^- &&= 11\,00\,10\,11\,00,\\
\mathbf{s} \bullet (110)^\infty &= (\mathbb{L}(\mathbf{s})^+\mathbf{s}\mathbf{s}^-)^\infty &&= (11\,01\,00)^\infty.
\end{aligned}$$

## Proposition (A. & Kong, 2023)

*$(L^*, \bullet)$ is a non-Abelian semi-group. In particular, $L^*$ is closed under $\bullet$ and $\bullet$ is associative.*

Let

$$\Lambda(k) := \{\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k : \quad \mathbf{s}_i \in F^* \text{ for } i = 1, 2, \ldots, k\}$$

for $k = 1, 2, \ldots$; and

$$\Lambda := \bigcup_{k=1}^{\infty} \Lambda(k).$$

Then

$$F^* \subsetneqq \Lambda \subsetneqq L^*.$$

The set $\Lambda$ will provide our basic bricks for the critical values $\tau(\beta)$.

# Farey intervals and basic intervals

For each word $\mathbf{s} \in \Lambda$ we define special bases $\beta_\ell, \beta_r$ and $\beta_*$ by

$$(\mathbb{L}(\mathbf{s})^\infty)_{\beta_\ell} = 1, \qquad (\mathbb{L}(\mathbf{s})^+\mathbf{s}^\infty)_{\beta_r} = (\mathbf{s} \bullet 1^\infty)_{\beta_r} = 1,$$
$$(\mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_{\beta_*} = (\mathbf{s} \bullet 10^\infty)_{\beta_*} = 1.$$

We call $J^s := [\beta_\ell, \beta_r]$ the Lyndon interval generated by $\mathbf{s}$.
Particularly, if $\mathbf{s} \in F^*$, we call $J^{\mathbf{s}}$ a Farey interval.

Furthermore, we call $I^s := [\beta_\ell, \beta_*]$ a basic interval generated by $\mathbf{s}$.
Note that $\beta_* < \beta_r$, so $I^s \subset J^s$.

Define the exceptional set

$$E := (1, 2] \setminus \bigcup_{\mathbf{s} \in F^*} J^{\mathbf{s}}.$$

## Farey intervals and the exceptional set

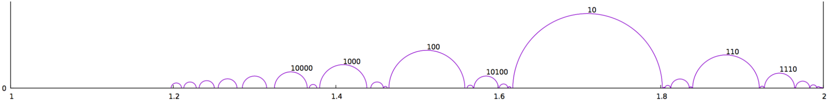



Figure: The graph of the Farey intervals $J^{\mathbf{s}}, \mathbf{s} \in F^*$.

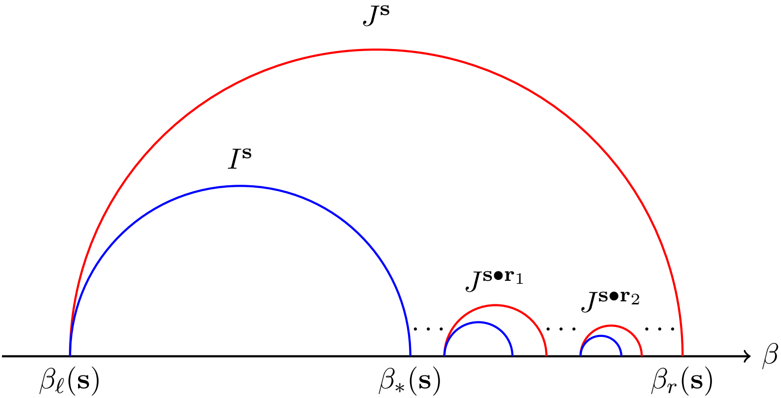
(Source: *Ergodic Theory Dynam. Systems* **40** (2020), p. 2495)

### Proposition (Kalle, Kong, Langeveld & Li, 2020)

*The Farey intervals $J^{\mathbf{s}}, \mathbf{s} \in F^*$ are pairwise disjoint, and the exceptional set $E = (1,2] \setminus \bigcup_{\mathbf{s} \in F^*} J^{\mathbf{s}}$ is uncountable with zero Hausdorff dimension. Furthermore,*

$$\tau(\beta) = 1 - \frac{1}{\beta} \quad \forall\, \beta \in E.$$

# Illustration of key intervals



## Basic intervals and critical values

Recall that $I^{\mathbf{s}} = [\beta_\ell, \beta_*]$, where

$$(\mathbb{L}(\mathbf{s})^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (\mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_{\beta_*} = 1.$$

### Theorem (A. & Kong, 2023)

*For any* $\mathbf{s} \in \Lambda$ *we have*

$$\tau(\beta) = (\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_\beta = (\mathbf{s} \bullet 0^\infty)_\beta \quad \forall \beta \in I^{\mathbf{s}}.$$

# Two examples

Example 1. Let $\mathbf{s} = 01 \in F^*$. The basic interval $I^{01} = [\beta_\ell, \beta_*] \approx [1.61803, 1.73867]$ satisfies

$$((10)^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (1100(10)^\infty)_{\beta_*} = 1.$$

In fact, $\beta_\ell = (1+\sqrt{5})/2$. We have

$$\tau(\beta) = (00(10)^\infty)_\beta = \frac{1}{\beta(\beta^2-1)} \qquad \forall \beta \in I^{01}.$$

Example 2. Take $\mathbf{s} = 000101 = 001 \bullet 01 \in \Lambda$. The basic interval $I^{\mathbf{s}} = [\beta_\ell, \beta_*] \approx [1.5385, 1.5526]$ satisfies

$$((101000)^\infty)_{\beta_\ell} = 1, \qquad (101001\,000100\,(101000)^\infty)_{\beta_*} = 1.$$

And $\tau(\beta) = (000100\,(101000)^\infty)_\beta$ for all $\beta \in I^{\mathbf{s}}$.

# Sketch of the proof

Take $\beta \in [\beta_\ell, \beta_*]$ and $t^* = (\mathbf{s}^- \mathbb{L}(\mathbf{s})^\infty)_\beta$. Then

$$\begin{aligned} \mathbf{K}_\beta(t^*) &= \{\mathbf{x} : b(t^*, \beta) \preccurlyeq \sigma^n(\mathbf{x}) \preccurlyeq \alpha(\beta)\ \forall n \geq 0\} \\ &\subset \left\{\mathbf{x} : \mathbf{s}^- \mathbb{L}(\mathbf{s})^\infty \preccurlyeq \sigma^n(\mathbf{x}) \preccurlyeq \mathbb{L}(\mathbf{s})^+ \mathbf{s}^- \mathbb{L}(\mathbf{s})^\infty\ \forall n \geq 0\right\} \end{aligned}$$

If $\mathbf{x}$ contains $\mathbb{L}(\mathbf{s})^+$, it must end in $\mathbb{L}(\mathbf{s})^+ \mathbf{s}^- \mathbb{L}(\mathbf{s})^\infty$.

Else, if $\mathbf{x}$ contains $\mathbf{s}^-$, it must end in $\mathbf{s}^- \mathbb{L}(\mathbf{s})^\infty$.

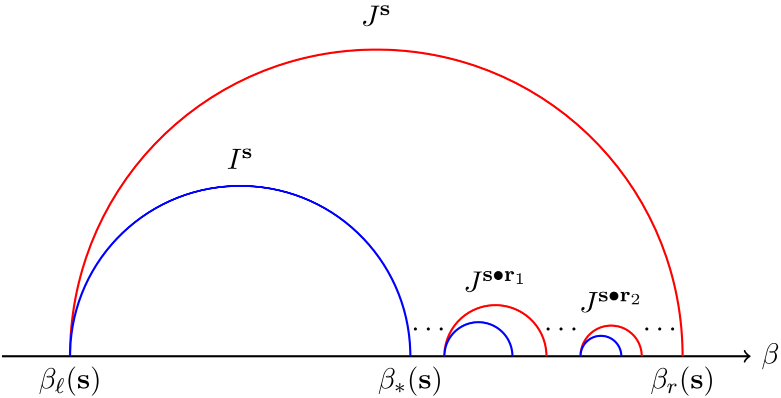
Else, $\mathbf{x} \in \Gamma(\mathbf{s})$, where

$$\Gamma(\mathbf{s}) := \{\mathbf{x} \in \{0,1\}^{\mathbb{N}} : \mathbf{s}^\infty \preccurlyeq \sigma^n(\mathbf{x}) \preccurlyeq \mathbb{L}(\mathbf{s})^\infty\ \forall n \geq 0\}.$$

We can prove that $\Gamma(\mathbf{s})$ is countable for $\mathbf{s} \in \Lambda$. Hence $\tau(\beta) \leq t^*$.

For "$\geq$", we argue similarly to the proof for $\beta = 2$.

# Illustration of key intervals



# Relative exceptional sets

## Proposition (A. & Kong, 2023)

(i) *For any* $\mathbf{s} \in \Lambda$ *the Lyndon intervals* $J^{\mathbf{s}\bullet\mathbf{r}}, \mathbf{r} \in F^*$ *are pairwise disjoint subsets of* $J^{\mathbf{s}} \setminus I^{\mathbf{s}}$*, and the relative exceptional set*

$$E^{\mathbf{s}} := (J^{\mathbf{s}} \setminus I^{\mathbf{s}}) \setminus \bigcup_{\mathbf{r} \in F^*} J^{\mathbf{s}\bullet\mathbf{r}}$$

*is uncountable with* $\dim_B E^{\mathbf{s}} = 0$.

(ii) *If* $\beta \in E^{\mathbf{s}}$*, then* $\alpha(\beta) = \mathbf{s} \bullet \alpha(\hat{\beta})$ *for some* $\hat{\beta} \in E$*, and*

$$\tau(\beta) = \left(\mathbf{s} \bullet b\Big(1 - \frac{1}{\hat{\beta}}, \hat{\beta}\Big)\right)_{\beta}.$$

(We say points in $E^{\mathbf{s}}$ are finitely renormalizable.)

## The infinitely renormalizable set

Define the infinitely renormalizable set by

$$E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{s}\in\Lambda(k)} J^{\mathbf{s}}.$$

Note that for each $\beta \in E_\infty$, there is a unique sequence $(\mathbf{s}_k) \subset F^*$, which we call the coding of $\beta$, such that

$$\{\beta\} = \bigcap_{k=1}^{\infty} J^{\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k}.$$

### Proposition (A. & Kong, 2023)

*$E_\infty$ is uncountable with $\dim_H E_\infty = 0$. Furthermore, if $\beta \in E_\infty$ with coding $(\mathbf{s}_i)$, then*

$$\tau(\beta) = \lim_{k\to\infty} (\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k 0^\infty)_\beta.$$

## A concrete example

Let $(\mathfrak{m}_i)_{i=0}^{\infty} = 01101001\ldots$ be the Thue-Morse sequence. Let $\beta$ be given by

$$(\mathfrak{m}_1\mathfrak{m}_2\ldots)_\beta = 1.$$

Then:

- $\beta$ is the Komornik-Loreti constant ($\beta \approx 1.787$);
- $\beta \in E_\infty$;
- $\beta$ is transcendental;
- $\tau(\beta) = \frac{2-\beta}{\beta-1} \approx 0.2703.$

(This can be generalized to compute $\tau(\beta)$ exactly for infinitely many $\beta$'s in $E_\infty$.)

Conjecture: Each $\beta \in E_\infty$ is transcendental.

# Extension to all $\beta > 1$

**Question:** How can we generalize all this to $\beta > 2$?

It appears that we must find the "right" way to generalize the Farey words to larger alphabets.

HOW ???

**Answer:** Just translate them!

# Extended Farey words

- Let $\theta(i) := i + 1$ for $i \in \mathbb{Z}$, and extend $\theta$ to finite words by homomorphism, e.g. $\theta(011) = 122$, etc.
- Let $\theta^k := \theta \circ \cdots \circ \theta$ ($k$ times).
- Define the set of extended Farey words by

$$F_e := \{\theta^k(\mathbf{s}) : \mathbf{s} \in F^* \cup \{1\}, k = 0, 1, 2, \ldots\}.$$

- Let $\Lambda_e(1) := F_e$, and for $k \geq 2$,

$$\Lambda_e(k) : = \{\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k : \mathbf{s}_1 \in F_e \text{ and } \mathbf{s}_2, \ldots, \mathbf{s}_k \in F^*\}.$$

- Put $\Lambda_e := \bigcup_{k=1}^{\infty} \Lambda_e(k)$.

# Extended Farey words, examples

## Example

For instance, $\Lambda_e$ contains the "new" words

- $1$;
- $2 = \theta(1)$;
- $1 \bullet 01 = 1^{-}\mathbb{L}(1)^{+} = 02$;
- $2 \bullet 01 = 2^{-}\mathbb{L}(2)^{+} = 13 = \theta(02) = \theta(1 \bullet 01)$;
- $1 \bullet 011 = 021$;
- $12 \bullet 001 = \theta(01 \bullet 001) = \theta(00\,10\,11) = 11\,21\,22$;

**Note:**

1. Words in $\Lambda_e$ can use 3 different letters!
2. $\theta$ commutes with $\bullet$.

# Generalizing the relevant sets

For $\mathbf{s} \in \Lambda_e$, again define intervals $I^{\mathbf{s}} = [\beta_\ell, \beta_*]$ and $J^{\mathbf{s}} = [\beta_\ell, \beta_r]$ by

$$(\mathbb{L}(\mathbf{s})^\infty)_{\beta_\ell} = 1, \qquad (\mathbb{L}(\mathbf{s})^+\mathbf{s}^\infty)_{\beta_r} = 1,$$
$$(\mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_{\beta_*} = 1.$$

We redefine $E$ as

$$E := (1, \infty) \setminus \bigcup_{\mathbf{s} \in F_e} J^{\mathbf{s}},$$

and again define

$$E^{\mathbf{s}} := (J^{\mathbf{s}} \setminus I^{\mathbf{s}}) \setminus \bigcup_{\mathbf{r} \in F^*} J^{\mathbf{s} \bullet \mathbf{r}}, \qquad \mathbf{s} \in \Lambda_e$$

and

$$E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{s} \in \Lambda_e(k)} J^{\mathbf{s}}.$$

## Generalizing the critical value theorems

**Theorem**

1. *If* $\beta \in E$, *then* $\tau(\beta) = 1 - (1/\beta)$.
2. *If* $\beta \in I^{\mathbf{s}}$ *for* $\mathbf{s} \in \Lambda_e$, *then*

$$\tau(\beta) = (\mathbf{s}^{-}\mathbb{L}(\mathbf{s})^{\infty})_{\beta}.$$

3. *If* $\beta \in E^{\mathbf{s}}$ *for* $\mathbf{s} \in \Lambda_e$, *then* $\alpha(\beta) = \mathbf{s} \bullet \alpha(\hat{\beta})$ *for some* $\hat{\beta} \in E$, *and*

$$\tau(\beta) = \left(\mathbf{s} \bullet b\Big(1 - \frac{1}{\hat{\beta}}, \hat{\beta}\Big)\right)_{\beta}.$$

4. *If* $\beta \in E_{\infty}$ *with "coding"* $\mathbf{s}_1, \mathbf{s}_2, \ldots$, *then*

$$\tau(\beta) = \lim_{k \to \infty} (\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k 0^{\infty})_{\beta}.$$

# Examples

Example 1. Let $\mathbf{s} = 1 \in F_e$. Then $I^{\mathbf{s}} = [\beta_\ell, \beta_*] = [2, \beta_*]$ satisfies

$$(\mathbb{L}(\mathbf{s})^\infty)_{\beta_\ell} = (1^\infty)_{\beta_\ell} = 1$$

and

$$(\mathbb{L}(\mathbf{s})^+\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_{\beta_*} = (201^\infty)_{\beta_*} = 1,$$

and we have

$$\tau(\beta) = (\mathbf{s}^-\mathbb{L}(\mathbf{s})^\infty)_\beta = (01^\infty)_\beta = \frac{1}{\beta(\beta-1)} \qquad \forall \beta \in I^1.$$

## Another Komornik-Loreti constant

Example 2. Let $\beta \in E_\infty$ with coding $(1, 01, 01, 01, \dots)$. Then

$$(2102\,0121\,0120\,2102\dots)_\beta = 1$$

and $\beta \approx 2.53595$ is the Komornik-Loreti constant for the alphabet $\{0, 1, 2\}$, i.e. the smallest base in which $1$ has a unique expansion over $\{0, 1, 2\}$. We have

$$\tau(\beta) = \frac{3 - \beta}{\beta - 1}.$$

(In fact, all of the Komornik-Loreti constants belong to $E_\infty$.)

## Global behavior of $\beta \mapsto \tau(\beta)$

Theorem (A. & Kong, 2023, 2024)

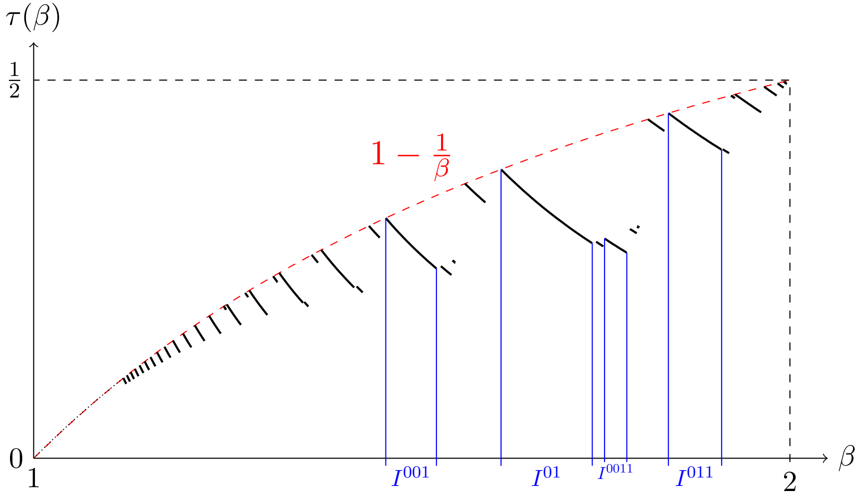
(i) *The function* $\tau : \beta \mapsto \tau(\beta)$ *is left continuous on* $(1, \infty)$ *with right-hand limits everywhere;*

(ii) $\tau$ *has no downward jumps;*

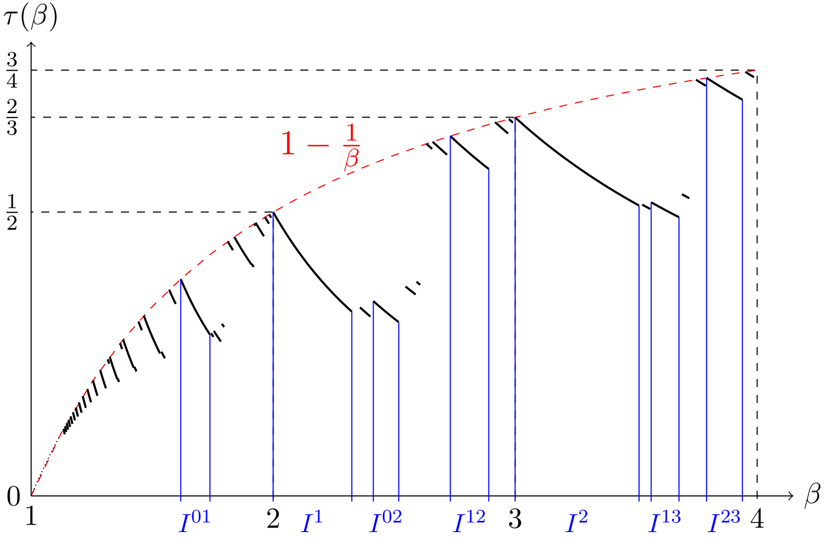
(iii) *The upward jumps of* $\tau$ *occur precisely at the points* $\beta_r(\mathbf{s}), \mathbf{s} \in \Lambda_e$, *given by*

$$(\mathbf{s} \bullet 1^\infty)_{\beta_r(\mathbf{s})} = 1.$$

(iv) *There is an open set* $O \subset (1, \infty)$ *with* $\dim_H((1, \infty) \setminus O) = 0$, *such that* $\tau$ *is real-analytic, convex and strictly decreasing on each connected component of* $O$.




The graph of the critical value function $\tau(\beta)$ for $\beta \in (1, 2]$.




The graph of $\tau(\beta)$ for $\beta \in (1, 4]$

# Bifurcation sets

We define two bifurcation sets:

$$\mathscr{E}_\beta := \{t \in [0,1) : K_\beta(t) \neq K_\beta(t+\varepsilon)\ \forall \varepsilon > 0\}$$

and

$$\mathscr{B}_\beta := \{t \in [0,1) : \eta_\beta(t) > \eta_\beta(t+\varepsilon)\ \forall \varepsilon > 0\},$$

where $\eta_\beta(t) = \dim_H K_\beta(t)$.

Obviously, $\mathscr{B}_\beta \subset \mathscr{E}_\beta$.

- Urbański (1987): $\mathscr{B}_2 = \mathscr{E}_2$
- Baker & Kong (2020): $\mathscr{B}_\beta = \mathscr{E}_\beta$ when $\beta$ is a multinacci number (root of $x^n = x^{n-1} + \cdots + x + 1$ for $n \geq 2$).

# Bifurcation set comparison

For general $\beta$, the situation is quite different...

**Theorem (A. & Kong, 2023)**

(i) $\dim_H(\mathscr{E}_\beta \backslash \mathscr{B}_\beta) > 0$ *for Lebesgue-almost every* $\beta \in (1, 2]$.

(ii) *For each* $k \in \{0, 1, 2, \dots\} \cup \{\aleph_0\}$, *there are infinitely many values of* $\beta$ *such that* $|\mathscr{E}_\beta \backslash \mathscr{B}_\beta| = k$.

(iii) *There is no* $\beta \in (1, 2]$ *such that* $\mathscr{E}_\beta \backslash \mathscr{B}_\beta$ *is uncountable but of zero Hausdorff dimension.*

## Local Hausdorff dimension of the bifurcation sets

Urbański (1987) proved that

$$\dim_H \left(\mathscr{E}_2 \cap [t,1)\right) = \dim_H K_2(t) \qquad \forall t \in (0,1)$$

and deduced from this the local dimension

$$\lim_{\varepsilon \searrow 0} \dim_H \left(\mathscr{E}_2 \cap (t-\varepsilon, t+\varepsilon)\right) = \dim_H K_2(t) \qquad \forall\, t \in \mathscr{E}_2.$$

Kalle, Kong, Langeveld & Li (2020) conjectured that

$$\dim_H \left(\mathscr{E}_\beta \cap [t,1)\right) = \dim_H K_\beta(t) \qquad \forall t \in (0,1).$$

This was proved by Baker & Kong (2020) for $\beta$ a multinacci number.

**Theorem (A. & Kong, 2023)**

*The conjecture of Kalle et al. holds for all $\beta \in (1,2]$:*

$$\dim_H \left(\mathscr{E}_\beta \cap [t,1)\right) = \dim_H K_\beta(t) \qquad \forall t \in (0,1).$$

(The proof is rather involved...)

**Theorem (A. & Kong, 2023)**

(i) *For each $\beta \in (1,2]$ and $t \in \mathscr{B}_\beta$, we have*

$$\lim_{\varepsilon \searrow 0} \dim_H \left(\mathscr{E}_\beta \cap (t-\varepsilon, t+\varepsilon)\right) = \dim_H K_\beta(t).$$

(ii) *If $\alpha(\beta)$ is eventually periodic, then for each $t \in \mathscr{B}_\beta$,*

$$\lim_{\varepsilon \searrow 0} \dim_H \left(\mathscr{B}_\beta \cap (t-\varepsilon, t+\varepsilon)\right) = \dim_H K_\beta(t).$$

*(This holds for a countable but dense set of $\beta$'s.)*

# Thank you!