Addition Automata and Attractors of Digit Systems Corresponding to Expanding Rational Matrices

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$$x = d_0 + Ad_1 + \cdots + A^{k-1}d_{k-1} \tag{1}$$

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1. $\mathbb{Z}^n[A] =$ smallest nontrivial A-invariant $\mathbb{Z}$-submodule of $\mathbb{Q}^n$ containing $\mathbb{Z}^n$
2. $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$ is a finite abelian group (Jankauskas and Thuswaldner, 2022)

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Remark.

1. $(A, \mathcal{D})$ has the finiteness property
   $\Longrightarrow$ $\mathcal{D}$ contains a complete residue set of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$
2. If $0 \in \mathcal{D}$ then $(A, \mathcal{D})$ has the uniqueness property
   $\Longleftrightarrow$ $\mathcal{D}$ is a complete residue set of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$

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Remark. $\forall x \in \mathbb{Z}^n[A]$, $\forall k \in \mathbb{Z}^+$,

$$x = d_0 + Ad_1 + \cdots + A^{k-1}d_{k-1} + A^k\Phi^k(x)$$

where $d_i = \boldsymbol{d}\big(\Phi^i(x)\big)$ for all $i < k$

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3. $n = 1 \implies A = \frac{p}{q} \in \mathbb{Q}$ in lowest terms with $|p| > |q|$ (Akiyama, Frougny, and Sakarovitch 2008)

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where $\varepsilon \in \{1, -1\}$, $p(x) = x^2 + \alpha x + \beta \in \mathbb{Z}[x]$ is irreducible, and $\mathcal{D} = \left\{(0,0)^\top, (1,0)^\top, \dots, (|\beta| - 1, 0)^\top\right\}$

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Arithmetics of Matrix Digit Systems

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1. every $x \in \mathbb{Z}^n[A]$ corresponds to a unique infinite word of the form $\sigma = (b_{r-1} \cdots b_1 b_0)^\infty d_{k-1} \cdots d_1 d_0$ satisfying $(\sigma)_A = x$

$\therefore$ every $x \in \mathbb{Z}^n[A]$ can be written uniquely in the form

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$$=: (\sigma)_A,$$

where $\sigma = (b_{r-1} \cdots b_1 b_0)^\infty d_{k-1} \cdots d_1 d_0$ (A-adic representation of x)

Remark.

1. every $x \in \mathbb{Z}^n[A]$ corresponds to a unique infinite word of the form $\sigma = (b_{r-1} \cdots b_1 b_0)^\infty d_{k-1} \cdots d_1 d_0$ satisfying $(\sigma)_A = x$
2. $x = (0^\infty d_{k-1} \cdots d_1 d_0)_A$

   $\implies$ x can be identified with the finite word $d_{k-1} \cdots d_1 d_0$

Companion Matrix Base

Consider. $A = \begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix} \in \mathbb{Q}^{2\times 2}$ with the following assumptions:

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Proposition

$\mathcal{D} = \left\{(0,0)^\top, (1,0)^\top, \cdots, (|b|-1, 0)^\top\right\}$ *is a complete residue system of* $\mathbb{Z}^2[A]/A\mathbb{Z}^2[A]$.

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Given:

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- $\mathcal{D} := \left\{(0,0)^\top, (1,0)^\top, \ldots, (|b|-1, 0)^\top\right\}$
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Define the following operations on σ:

$$
\begin{aligned}
(\sigma^{\pm P})_A &= (\sigma)_A \pm (1,0)^\top \\
(\sigma^{\pm Q})_A &= (\sigma)_A \pm A(c,0)^\top \pm (a-c,0)^\top \\
(\sigma^{\pm R})_A &= (\sigma)_A \mp A(c,0)^\top \mp (a,0)^\top \\
(\sigma^{\pm S})_A &= (\sigma)_A \pm (c,0)^\top \\
(\sigma^{\pm T})_A &= (\sigma)_A \pm A(c,0)^\top \pm (a+c,0)^\top \\
(\sigma^{\pm U})_A &= (\sigma)_A \pm (0,1)^\top = (\sigma)_A \pm A(1,0)^\top
\end{aligned}
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$\therefore \pm U \xrightarrow{\nu|\nu} \pm P$

The case $0 < \alpha \leq \beta - 1$

Given:

- expanding matrix $A = \begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix} \in \mathbb{Q}^{2\times 2}$ with irreducible characteristic polynomial $p(x) = x^2 + \alpha x + \beta \in \mathbb{Q}[x]$
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Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < \alpha \le \beta - 1$

Then: $\mathcal{A}_\Phi = \{0\}$

Notation.

1. $(\nu, 0)^\top \in \mathcal{D}$ will be identified with ν
2. $\overline{f(\nu)}$: $f(\nu) \in \mathcal{D}$ and is taken to be a single digit in the expansion

Transition relations for $\nu \in \mathcal{D}$:

$$(\sigma\nu)^P = \begin{cases} \sigma\, \overline{\nu+1}, & 0 \leq \nu < b-1 \\ \sigma^R\, 0, & \nu = b-1 \end{cases}$$

$$(\sigma\nu)^{-P} = \begin{cases} \sigma\, \overline{\nu-1}, & 1 \leq \nu < b \\ \sigma^{-R}\, \overline{b-1}, & \nu = 0 \end{cases}$$

$$(\sigma\nu)^{\pm Q} = \begin{cases} \sigma^{\pm T}\, \overline{\nu \pm a \mp c \pm b}, & 0 \leq \nu \pm a \mp c \pm b < b \\ \sigma^{\pm S}\, \overline{\nu \pm a \mp c}, & 0 \leq \nu \pm a \mp c < b \\ \sigma^{\mp Q}\, \overline{\nu \pm a \mp c \mp b}, & 0 \leq \nu \pm a \mp c \mp b < b \end{cases}$$

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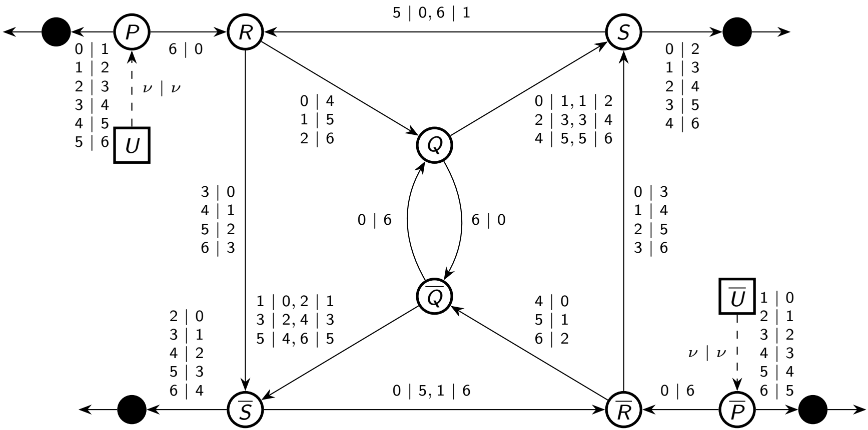
$$(\sigma\nu)^{\pm S} = \begin{cases} \sigma\,\overline{\nu \pm c}, & 0 \leq \nu \pm c < b \\ \sigma^{\pm R}\,\overline{\nu \pm c \mp b}, & 0 \leq \nu \pm c \mp b < b \end{cases}$$

$$(\sigma\nu)^{\pm T} = \begin{cases} \sigma^{\pm S}\,\overline{\nu \pm a \pm c}, & 0 \leq \nu \pm a \pm c < b \\ \sigma^{\mp Q}\,\overline{\nu \pm a \pm c \mp b}, & 0 \leq \nu \pm a \pm c \mp b < b \end{cases}$$

The case $0 < \alpha \leq \beta - 1$

Transition diagram for $\alpha = \frac{3}{2}$ and $\beta = \frac{7}{2}$



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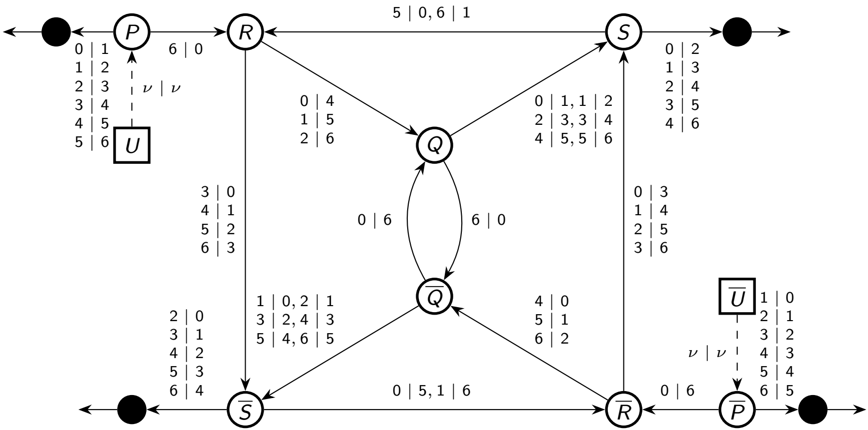

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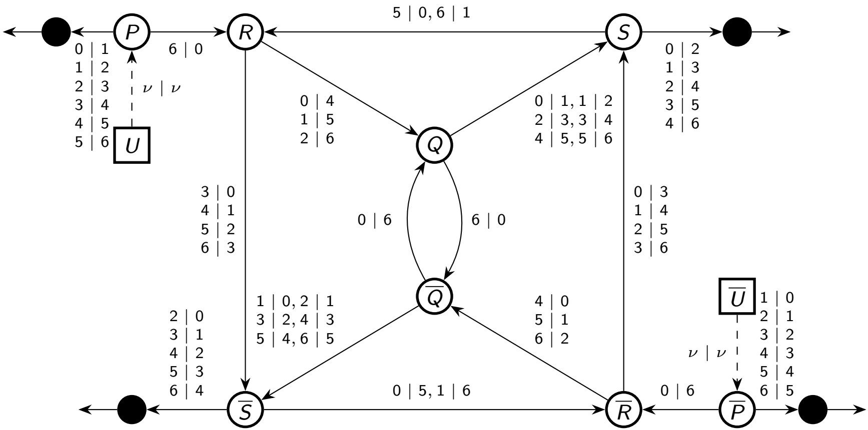

Remark. Any walk in $\mathcal{T}$ with label of the form $0 \mid d$ always leads to $\bullet$.

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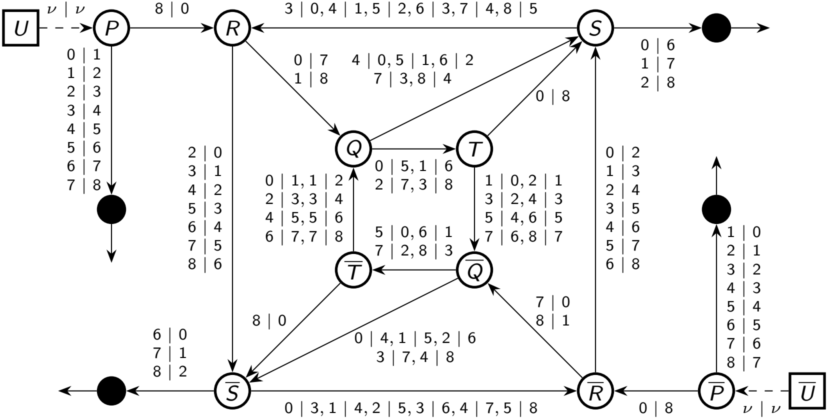
2. elements of $\mathcal{A}_\Phi$: arise from $\bullet$ or from loops when feeding a digit representation

Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):



e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

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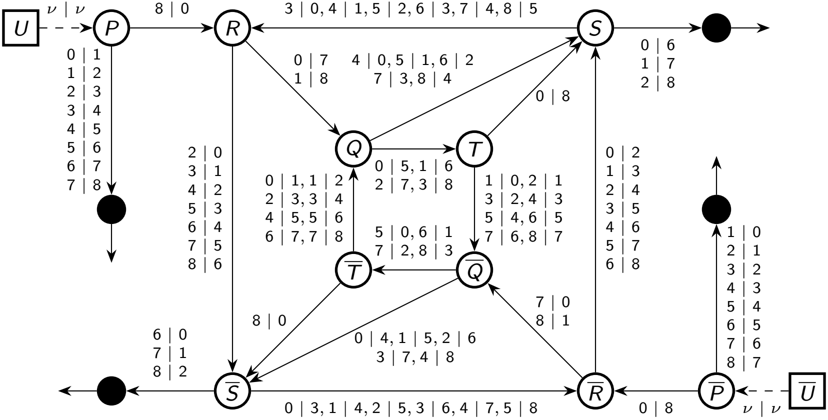

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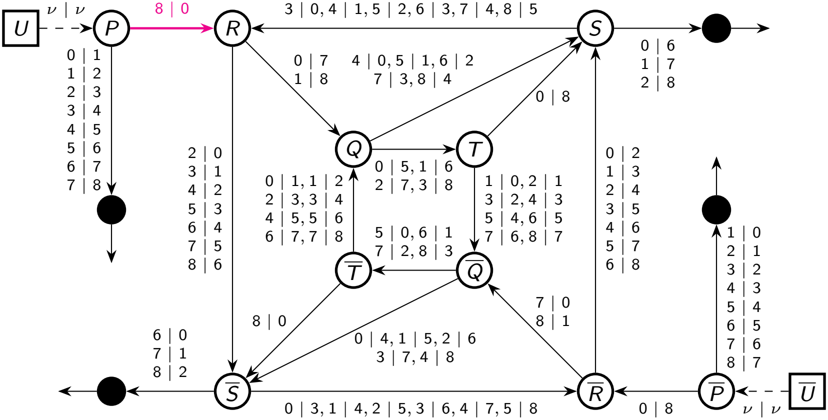

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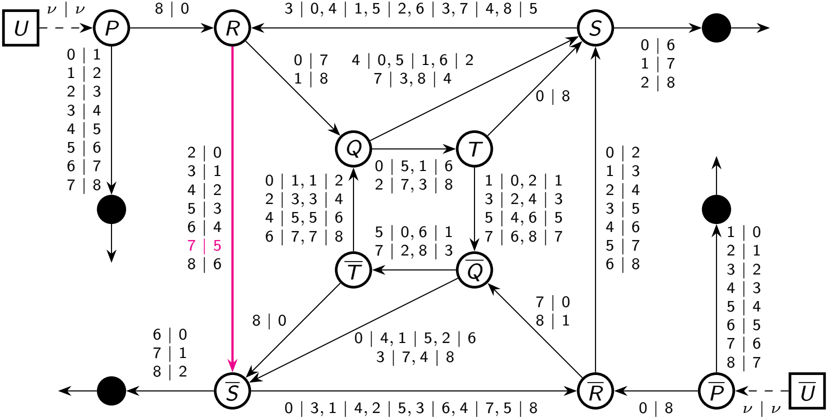

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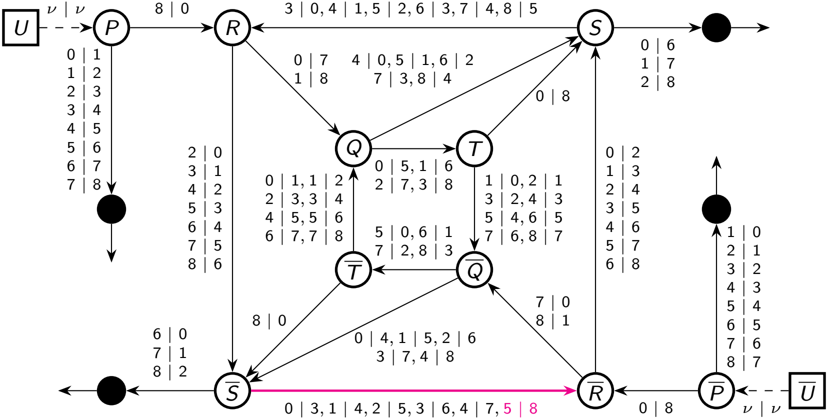

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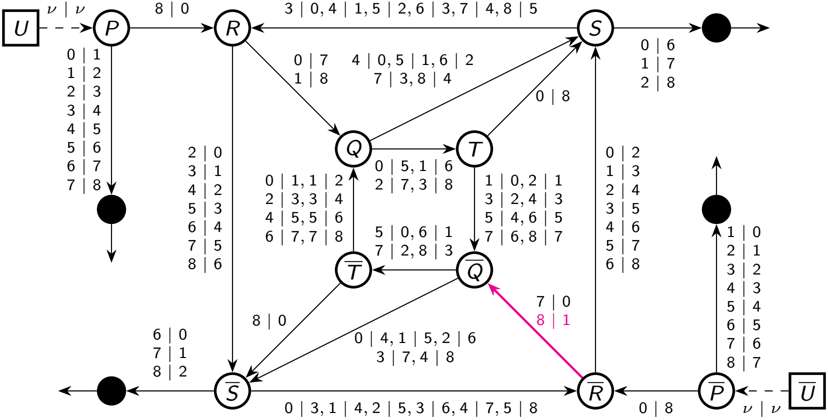
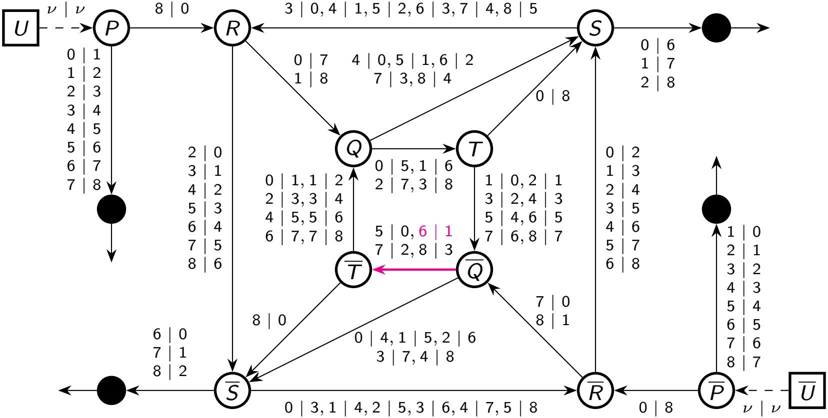


e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\Longrightarrow$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = \quad 1850$

The case $0 < \alpha \leq \beta - 1$

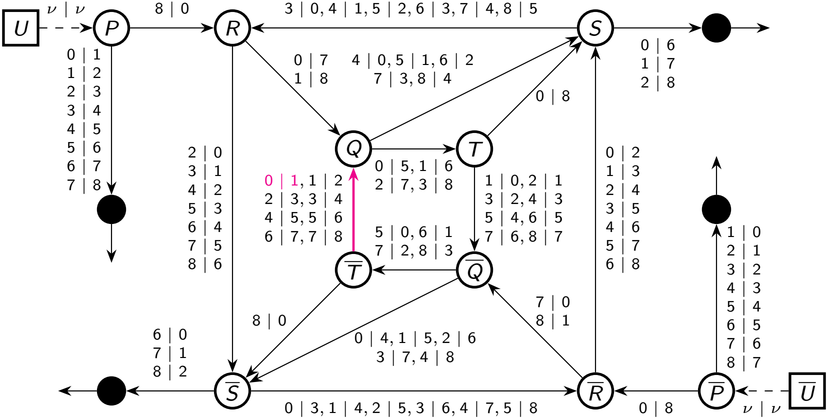
Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):



e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\Longrightarrow$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = 11850$

Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):

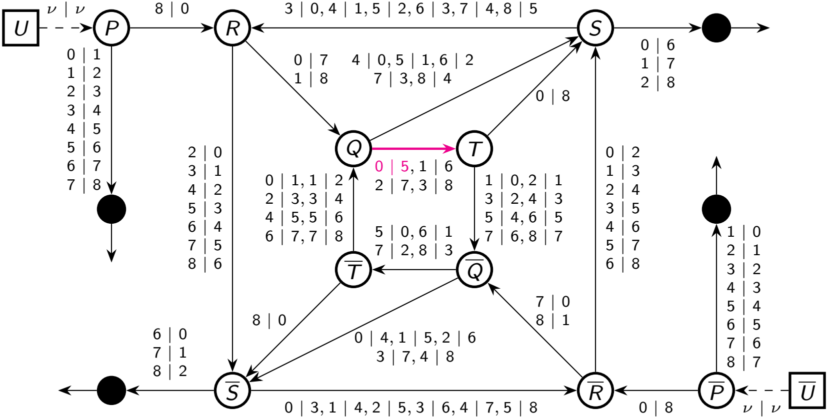


e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\Longrightarrow$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = \quad 111850$

The case $0 < \alpha \le \beta - 1$

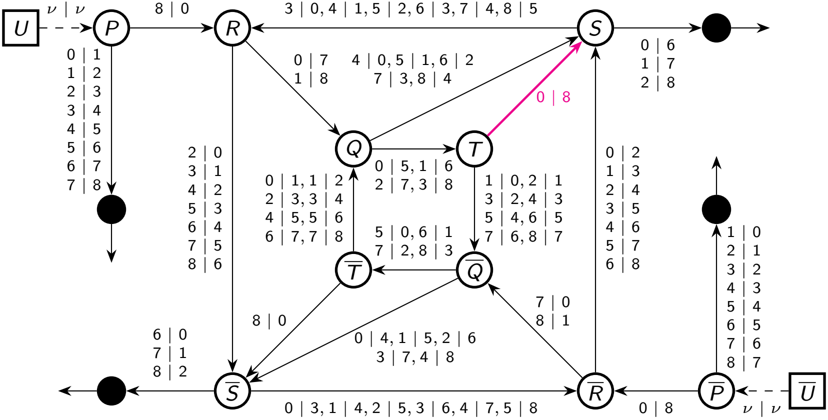
Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):



e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\implies$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = \quad 5111850$

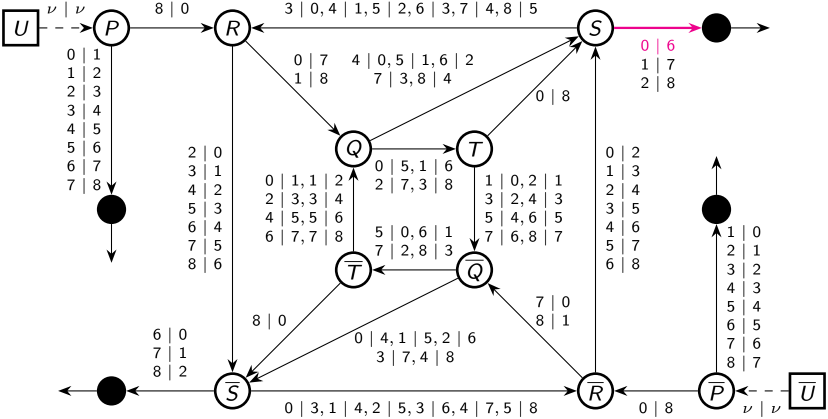
Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):



e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\Longrightarrow$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = 85111850$

Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):

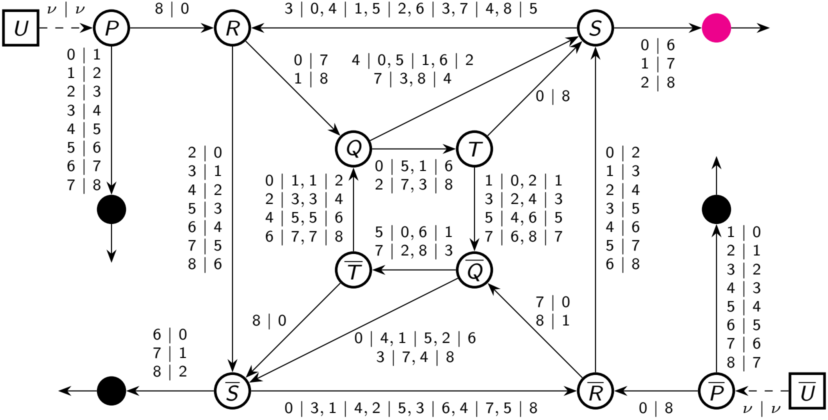


e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\Longrightarrow$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = 685111850$

The case $0 < \alpha \leq \beta - 1$

Transition diagram for $\alpha = \frac{1}{3}$ and $\beta = \frac{3}{2}$ (note: $0 < \alpha < 1$):



e.g.: $v = (17, 0)^\top$ with representation $\sigma = 68578$

$\implies$ representation of $(18, 0)^\top = (\sigma^P)_A$: $\sigma^P = 685111850$

Feed $(0,0)^\top$ (with representation $\sigma = 0^\infty$) into $\mathcal{T}$ at $\pm P$ and $\pm U$.

The case $0 < \alpha \leq \beta - 1$

Feed $(0,0)^\top$ (with representation $\sigma = 0^\infty$) into $\mathcal{T}$ at $\pm P$ and $\pm U$.

Then:

- $(\pm 1, 0)^\top = (\sigma^{\pm P})$ and $(0, \pm 1)^\top = (\sigma^{\pm U})_A$ have finite digit representations

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- $(m, n)^\top$ has a finite representation $\forall m, n \in \mathbb{Z}$

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- $(m,n)^\top$ has a finite representation $\forall m,n \in \mathbb{Z}$
- $A^k(m,n)^\top$ has a finite representation $\forall m,n \in \mathbb{Z}$, $\forall k \in \mathbb{N}$

The case $0 < \alpha \leq \beta - 1$

Feed $(0,0)^\top$ (with representation $\sigma = 0^\infty$) into $\mathcal{T}$ at $\pm P$ and $\pm U$.

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$\therefore$ every $v \in \mathbb{Z}^2[A]$ has a finite digit representation

The case $0 < \alpha \leq \beta - 1$

Feed $(0,0)^\top$ (with representation $\sigma = 0^\infty$) into $\mathcal{T}$ at $\pm P$ and $\pm U$.

Then:

- $(\pm 1, 0)^\top = (\sigma^{\pm P})$ and $(0, \pm 1)^\top = (\sigma^{\pm U})_A$ have finite digit representations
- $(m, n)^\top$ has a finite representation $\forall m, n \in \mathbb{Z}$
- $A^k(m, n)^\top$ has a finite representation $\forall m, n \in \mathbb{Z}$, $\forall k \in \mathbb{N}$

$\therefore$ every $v \in \mathbb{Z}^2[A]$ has a finite digit representation and thus $\mathcal{A}_\Phi = \{0\}$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

Then:

1. $\alpha \geq -1$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

Then:

1. $\alpha \geq -1 \implies \mathcal{A}_{\Phi} = \{0\}$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

Then:

1. $\alpha \geq -1 \implies \mathcal{A}_\Phi = \{0\}$
2. $\alpha < -1$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

$$\gamma = a + b + c$$

Then:

1. $\alpha \geq -1 \implies \mathcal{A}_{\Phi} = \{0\}$
2. $\alpha < -1$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

$$\gamma = a + b + c$$

$$K = \max\{\ell \in \mathbb{Z}^+ \mid \ell\gamma \leq b - 1\}$$

Then:

1. $\alpha \geq -1 \implies \mathcal{A}_\Phi = \{0\}$
2. $\alpha < -1$

The case $0 < -\alpha < \beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

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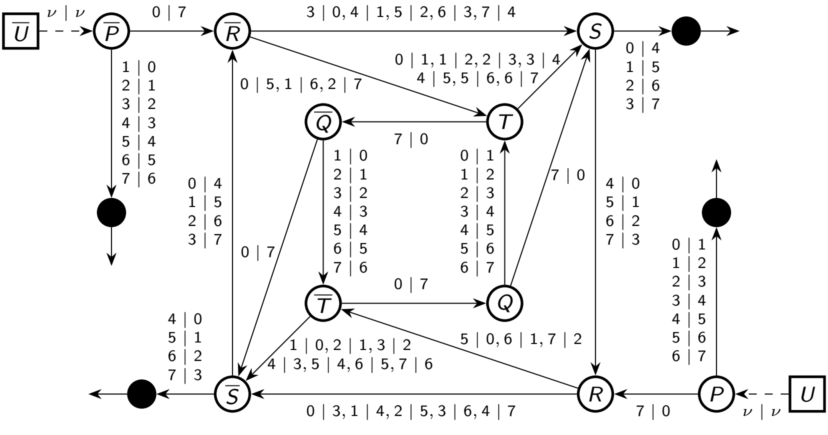
$$K = \max\{\ell \in \mathbb{Z}^+ \mid \ell\gamma \leq b - 1\}$$

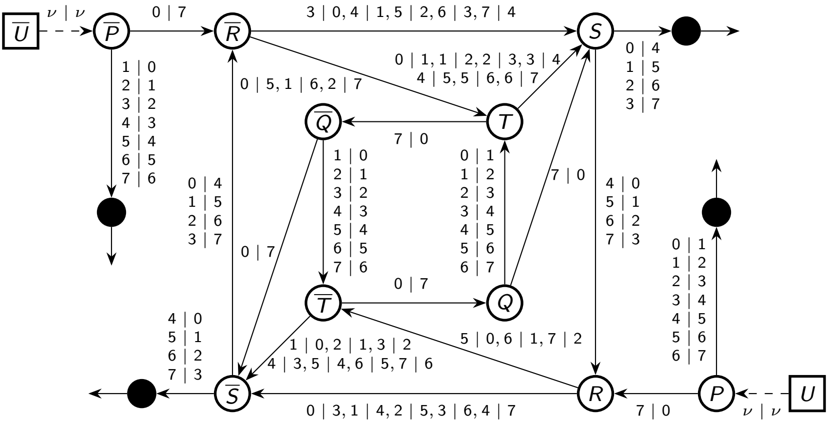
Then:

1. $\alpha \geq -1 \implies \mathcal{A}_\Phi = \{0\}$
2. $\alpha < -1 \implies \mathcal{A}_\Phi = \left\{(0,0)^\top, (a+c,c)^\top, \ldots, K(a+c,c)^\top\right\}$

Transition diagram for $\alpha = -\frac{3}{4}$ and $\beta = 2$ (note: $\alpha \geq -1$):

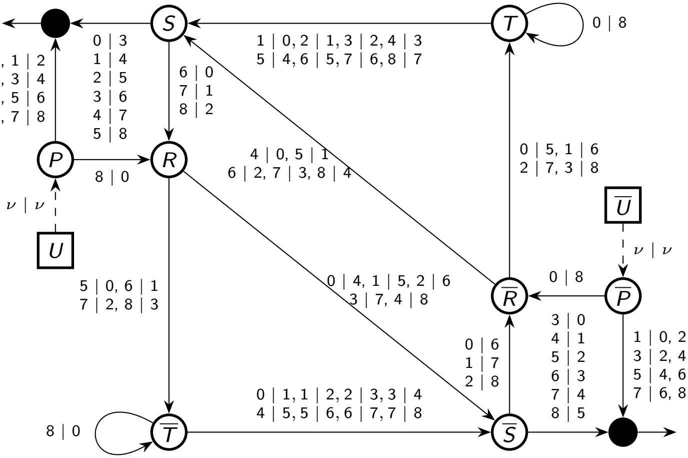


Transition diagram for $\alpha = -\frac{3}{4}$ and $\beta = 2$ (note: $\alpha \geq -1$):



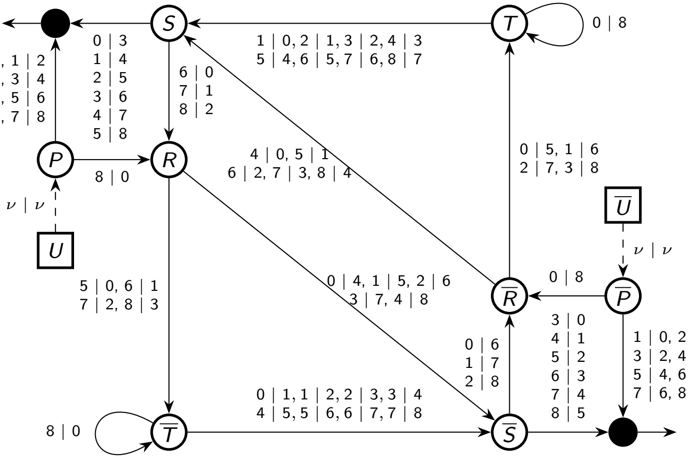
Remark. Any walk in $\mathcal{T}$ with only 0 as input always leads to $\bullet$.

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



The case $0 < -\alpha < \beta - 1$

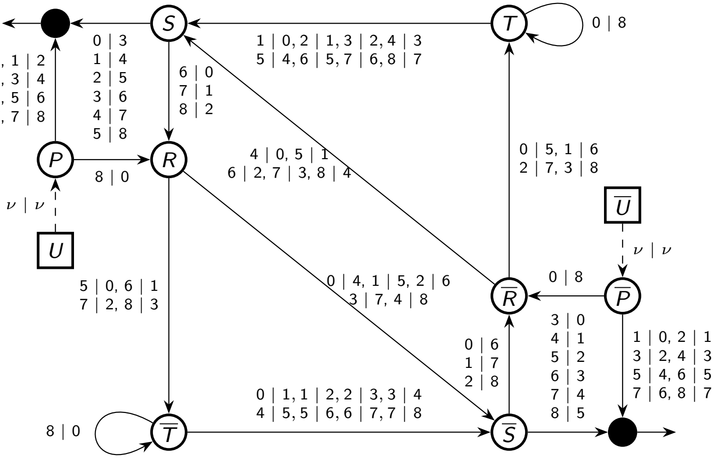
Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



Remark. Any walk in $\mathcal{T}$ with only 0 as input always leads to $\bullet$ or the self-loop at T.

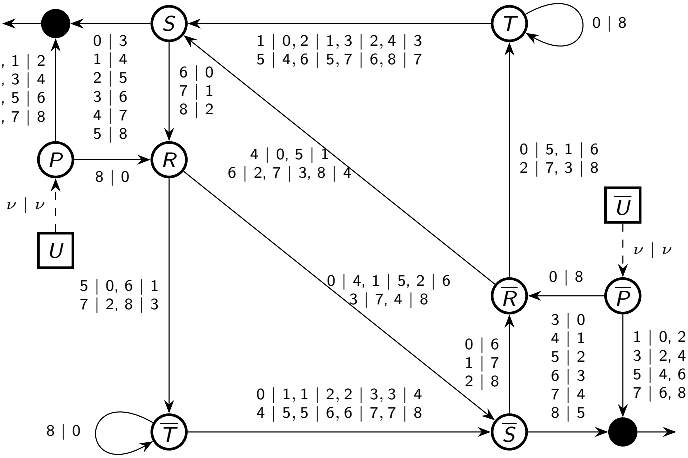
The case $0 < -\alpha < \beta - 1$

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



e.g.: $v = (8,2)^\top$ with representation $\sigma = 28$

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):

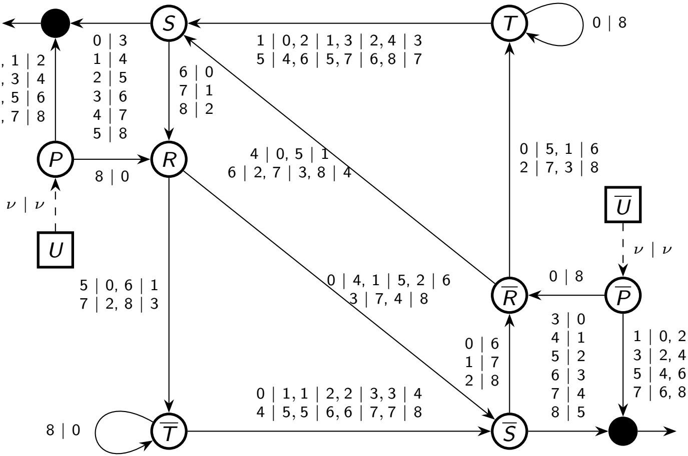


e.g.: $v = (8, 2)^\top$ with representation $\sigma = 28$

$\Longrightarrow$ representation of $(9, 2)^\top = (\sigma^P)_A$:

The case $0 < -\alpha < \beta - 1$

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



e.g.: $v = (8,2)^\top$ with representation $\sigma = 28$

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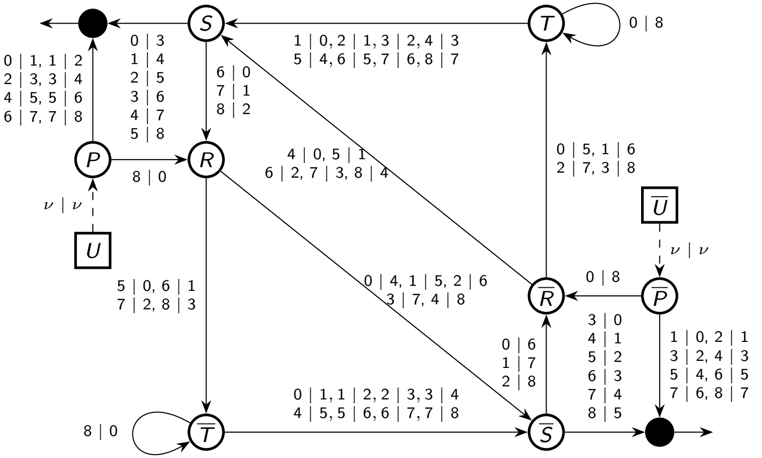
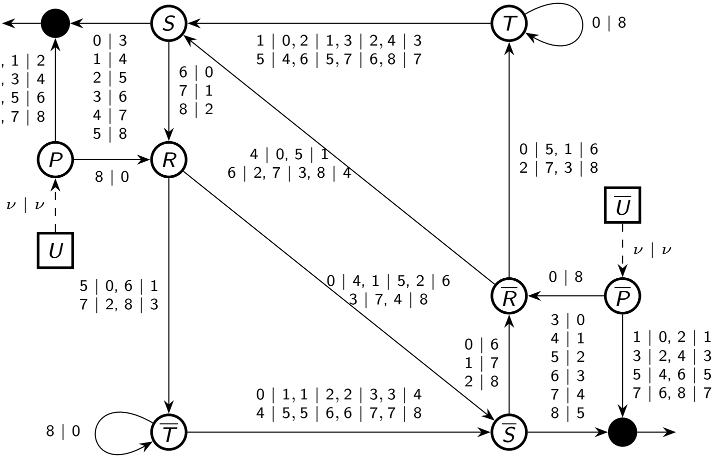
Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



e.g.: $v = (8, 2)^\top$ with representation $\sigma = 28$

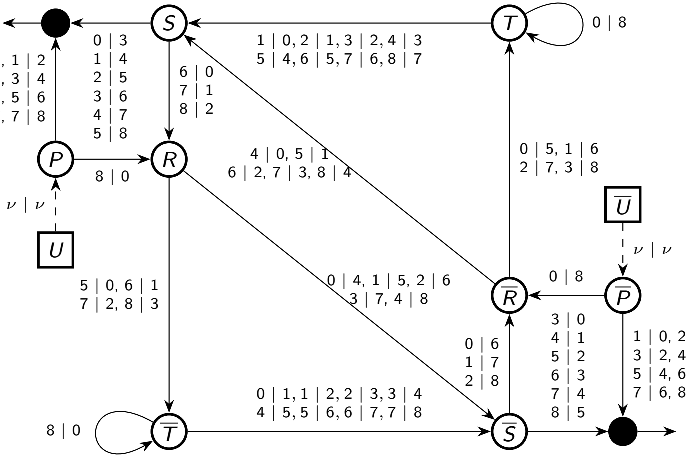
$\implies$ representation of $(9, 2)^\top = (\sigma^P)_A$: $\sigma^P = 8^\infty 5660$ (note: $\gamma = 8$)

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



$\Longrightarrow \; p := (8^\infty)_A \in \mathcal{A}_\Phi$

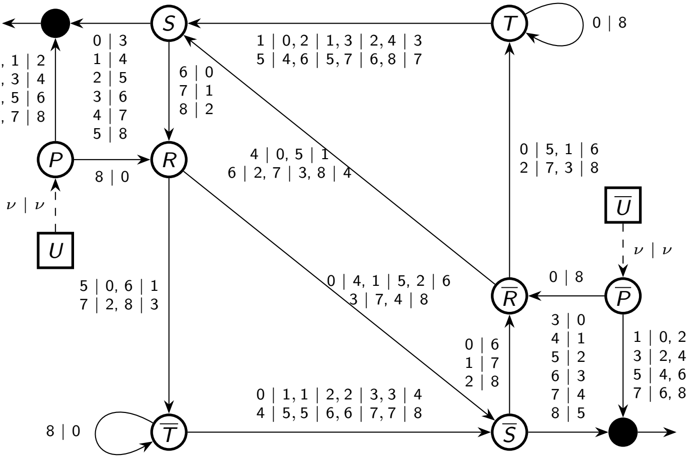
Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



$\implies \ p := (8^\infty)_A \in \mathcal{A}_\Phi$

$\implies \ p = (0,0)^\top + A(c,0)^\top + (a+c,0)^\top$

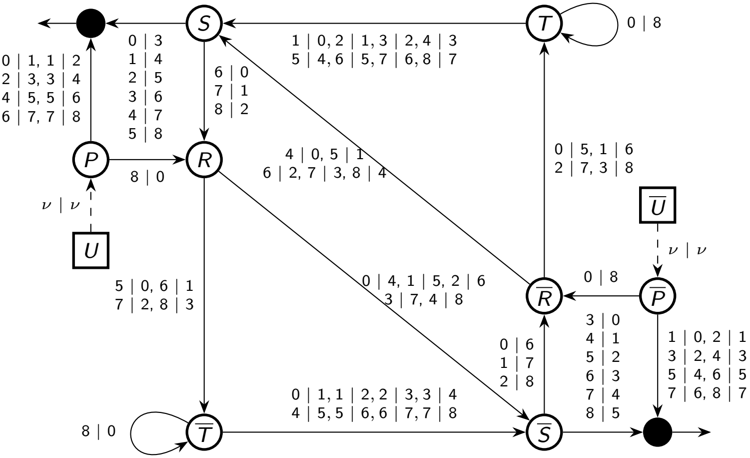
Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



$\implies$ $p := (8^\infty)_A \in \mathcal{A}_\Phi$

$\implies$ $p = (0,0)^\top + A(c,0)^\top + (a+c,0)^\top = (a+c,c)^\top$

Transition diagram for $\alpha = -\frac{4}{3}$ and $\beta = 3$ (note: $\alpha < -1$):



$$\Longrightarrow \quad p := (8^\infty)_A \in \mathcal{A}_\Phi$$

$$\Longrightarrow \quad p = (0,0)^\top + A(c,0)^\top + (a+c,0)^\top = (a+c,c)^\top = (-1,3)^\top$$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < -\beta - 1$

Then: $\mathcal{A}_\Phi$ *contains* $\{(0,0)^\top, -(a,c)^\top, -(c,0)^\top\}$

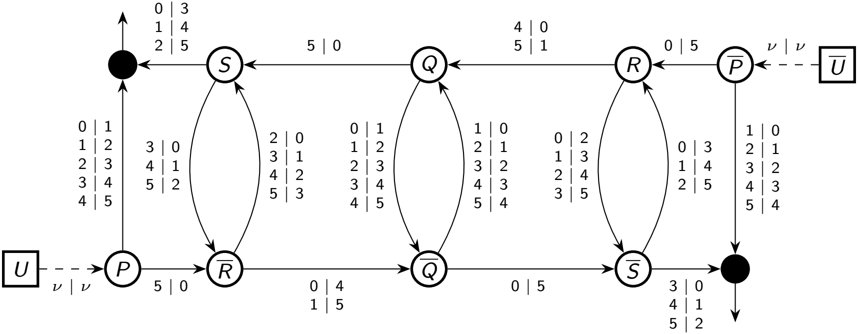
Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < -\beta - 1$

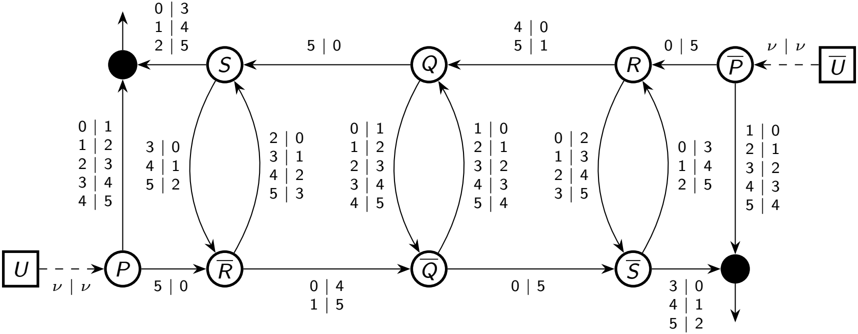
Then: $\mathcal{A}_\Phi$ *contains* $\{(0,0)^\top, -(a,c)^\top, -(c,0)^\top\}$

Remark. $\mathcal{A}_\Phi$ may contain other points depending on α and β.

Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:

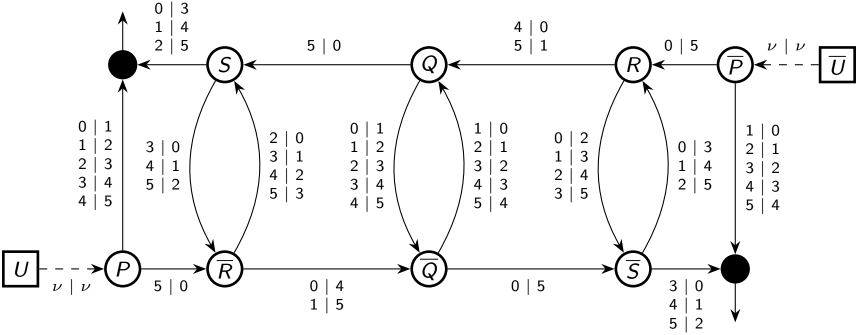


Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:



Remark. Any walk in $\mathcal{T}$ with only 0 as input leads to one of the following:

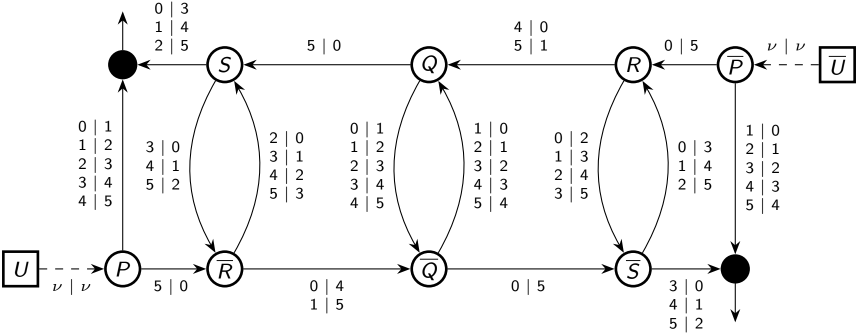
Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:



Remark. Any walk in $\mathcal{T}$ with only 0 as input leads to one of the following:

1. the terminal state $\bullet$

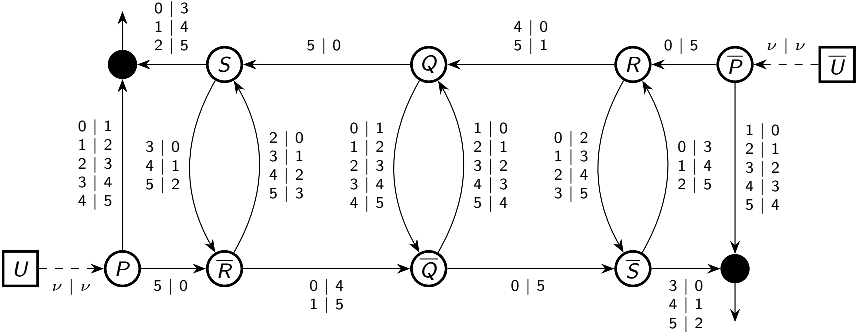
Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:



Remark. Any walk in $\mathcal{T}$ with only 0 as input leads to one of the following:

1. the terminal state $\bullet$
2. the loop $R \to -S \to R$

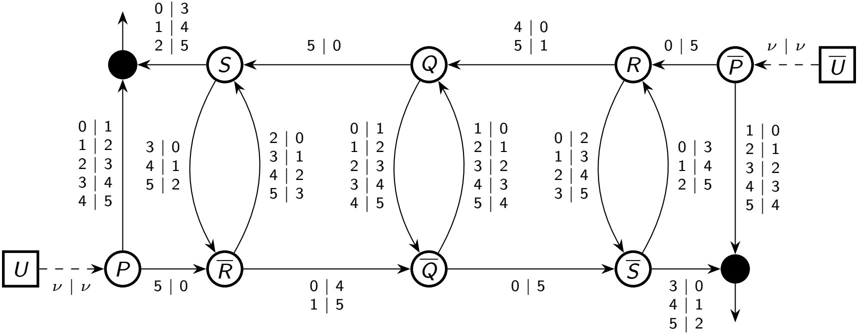
Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:



e.g.: $v = (7, -3)^\top$ with representation $\sigma = (32)^\infty 05$

$\Longrightarrow$ representation of $v + (1,0)^\top = (\sigma^P)_A$ is $\sigma^P = (41)^\infty 40$

Transition diagram for $\alpha = -\frac{2}{3}$ and $\beta = -2$:



e.g.: $v = (7, -3)^\top$ with representation $\sigma = (32)^\infty 05$

$\implies$ representation of $v + (1,0)^\top = (\sigma^P)_A$ is $\sigma^P = (41)^\infty 40$
and $p = (41)^\infty = (1,6)^\top \in \mathcal{A}_\Phi$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

$$\gamma = -a - b - c$$

$$K = \max\{\ell \in \mathbb{Z}^+ \mid \ell\gamma \leq -b - 1\}$$

The case $0 < \alpha < -\beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

$$\gamma = -a - b - c$$

$$K = \max\{\ell \in \mathbb{Z}^+ \mid \ell\gamma \leq -b - 1\}$$

Then: $\mathcal{A}_\Phi$ *contains* $\left\{(0,0)^\top, -(a+c,c)^\top, \ldots, -K(a+c,c)^\top\right\}$

The case $0 < \alpha < -\beta - 1$

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

Given: $0 < -\alpha < \beta - 1$

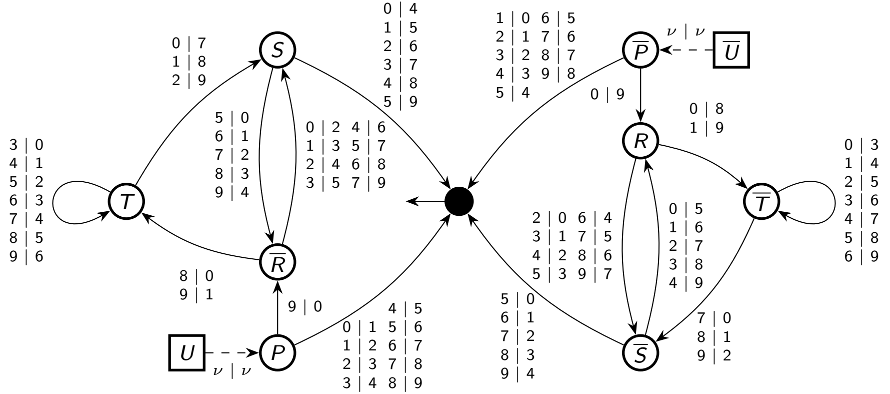
$$\gamma = -a - b - c$$

$$K = \max\{\ell \in \mathbb{Z}^+ \mid \ell\gamma \leq -b - 1\}$$

Then: $\mathcal{A}_\Phi$ *contains* $\{(0,0)^\top, -(a+c,c)^\top, \ldots, -K(a+c,c)^\top\}$

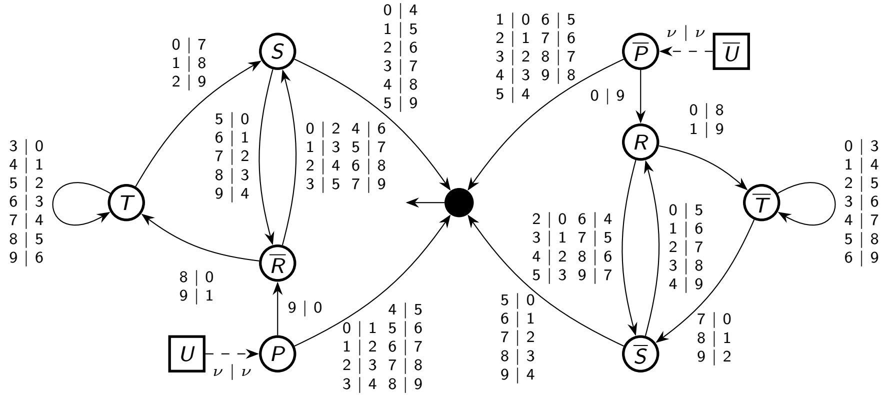
Remark. $\mathcal{A}_\Phi$ may contain other points depending on α and β.

Transition diagram for when $\alpha = \frac{2}{5}$ and $\beta = -2$:



The case $0 < \alpha < -\beta - 1$

Transition diagram for when $\alpha = \frac{2}{5}$ and $\beta = -2$:



Remark. Any walk in $\mathcal{T}$ with only 0 as input may lead to either the terminal state $\bullet$ or the self-loop at $-T$

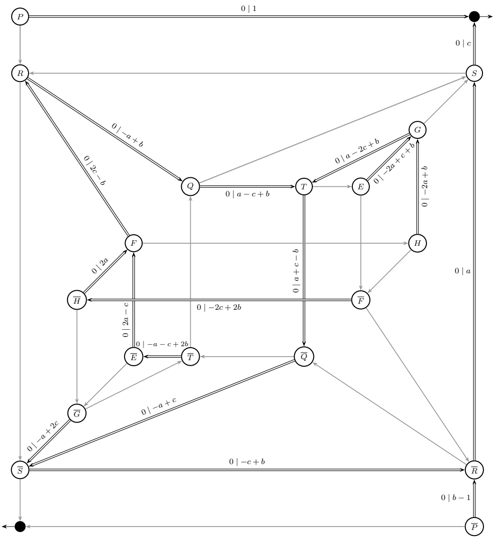
Additional Cases

Theorem (C.-Loquias-Thuswaldner, preprint, 2025)

1. $\alpha > 1$ *and* $\beta - 1 \leq \alpha \leq \beta \implies \mathcal{A}_{\Phi} = \{0\}$
2. $0 < \alpha \leq 1$ *and* $0 < \beta - 1 < \alpha < \beta$
   $\implies \mathcal{A}_{\Phi} = \{0\}$ *provided one of the following holds:*
   1. $\beta > 2\alpha$ *and* $\alpha + \beta > 2$
   2. $\beta \leq 2\alpha$ *and* $\alpha + 2 < 2\beta$

Additional Cases

Transition diagram for Case 2.1:



Additional Cases

Transition diagram for Case 2.2: