Construction of boundaries of tiles associated with nearest integer complex continued fractions over imaginary quadratic fields

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Numeration and Substitution 2025

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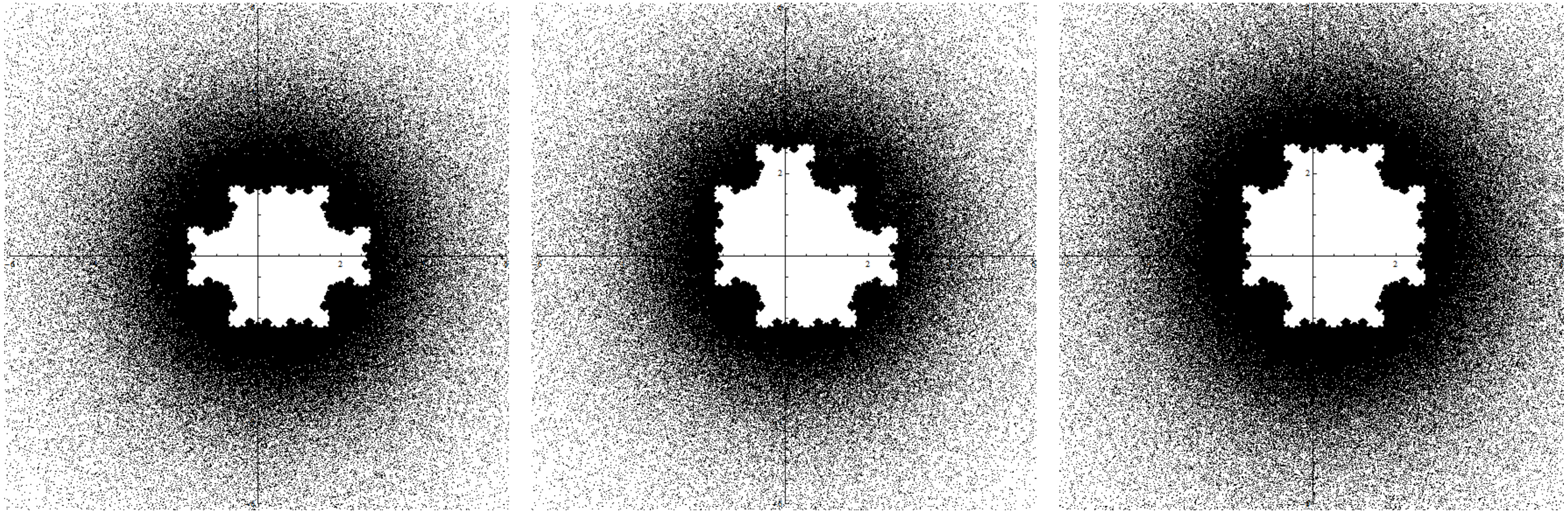
S. Ito, H. Nakada, R. Natsui, and H.E
"On the construction of the natural extension of the Hurwitz complex continued fraction map",
Monatsh. Math. 188 (2019), no. 1, 37-86.

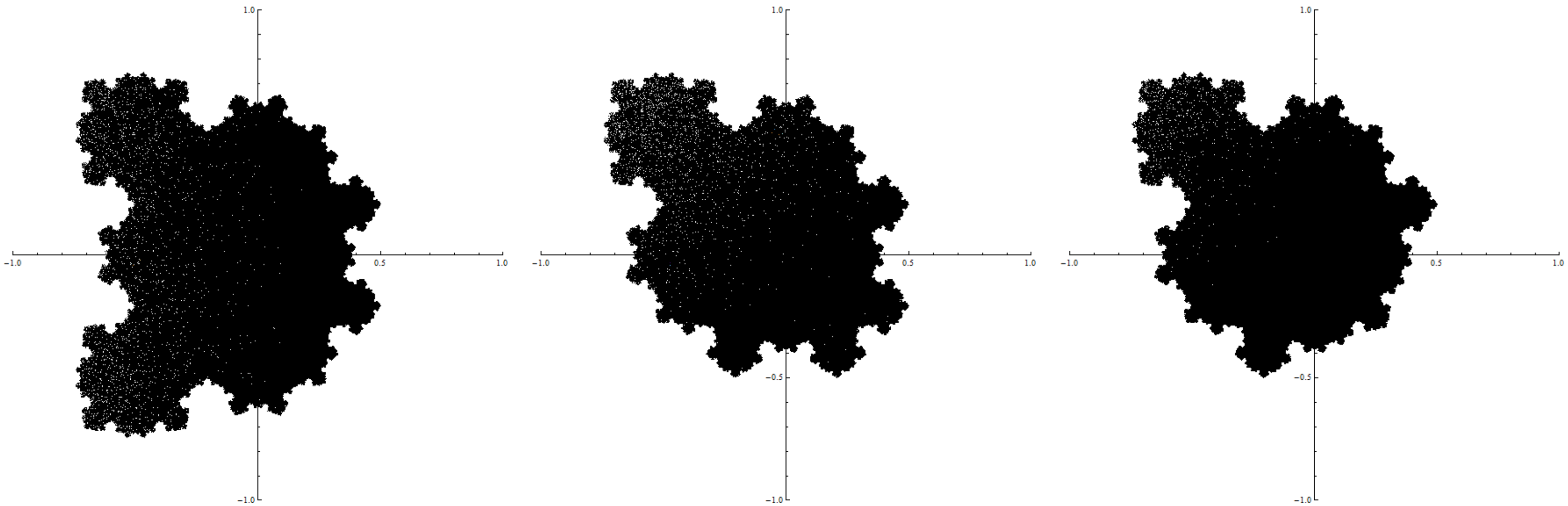
- In this paper, the nearest integer complex continued fraction map associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-1})$ is considered, and its natural extension is constructed.

As the domain of the second component of the natural extension, tilings are obtained.

Tilings



by 3 prototiles:



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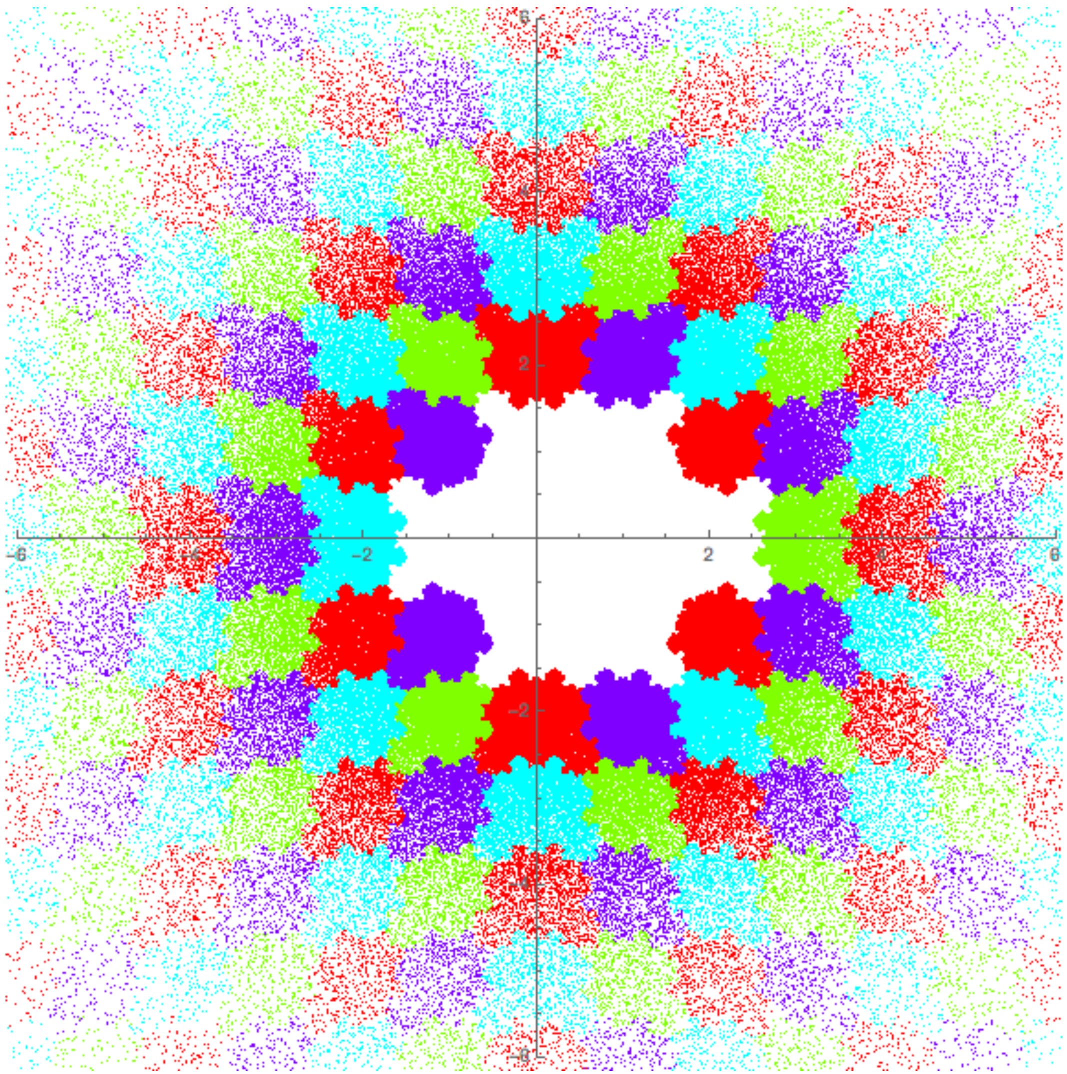
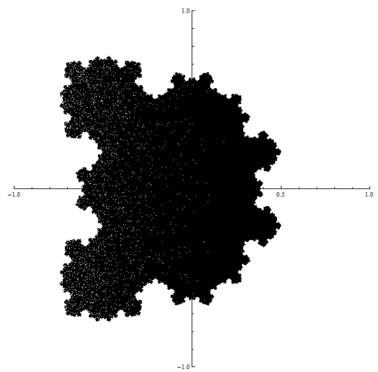
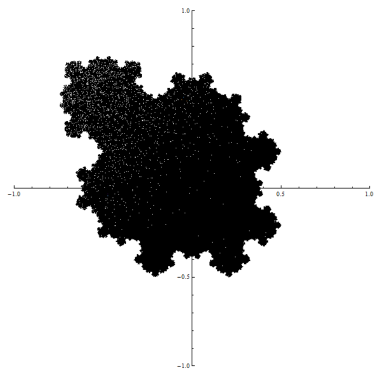
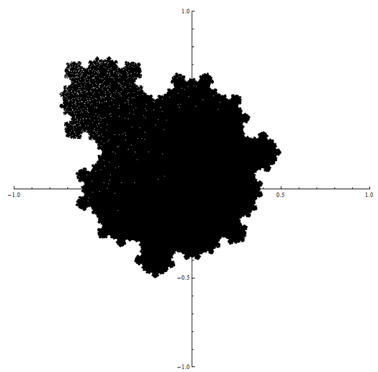
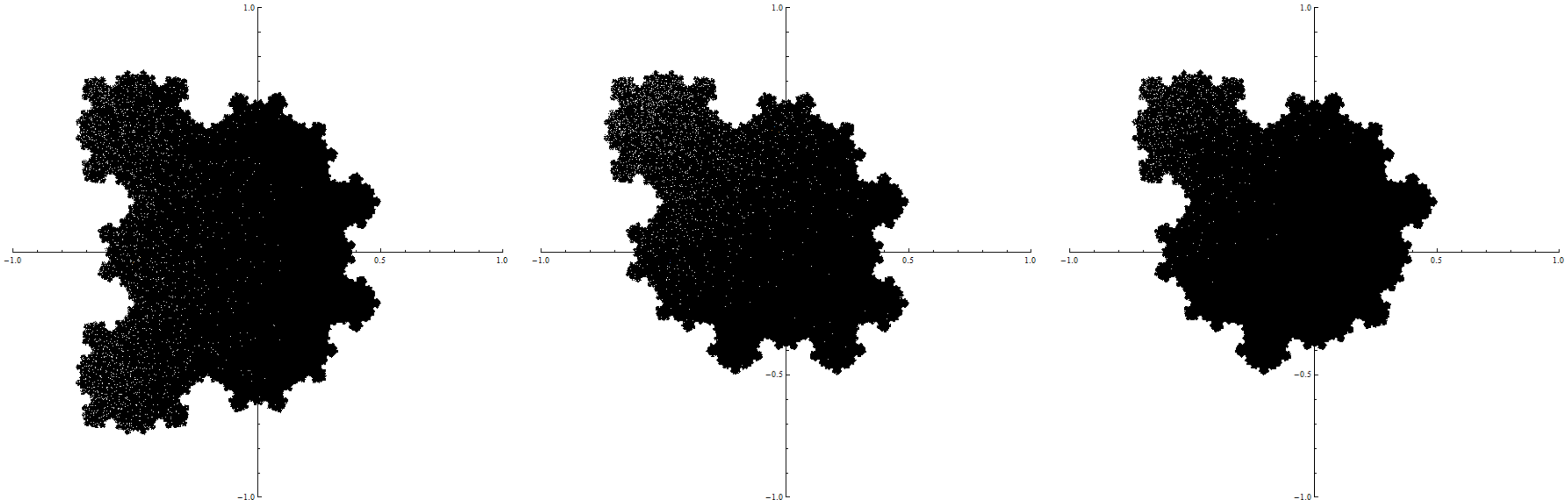








Aim



The boundaries of these prototiles look like " fractal curves".

How do we construct the boundaries?

Hurwitz complex CF map over $\mathbb{Q}(\sqrt{-1})$

$\mathfrak{o}$ is the set of algebraic integers of $\mathbb{Q}(\sqrt{-1})$:

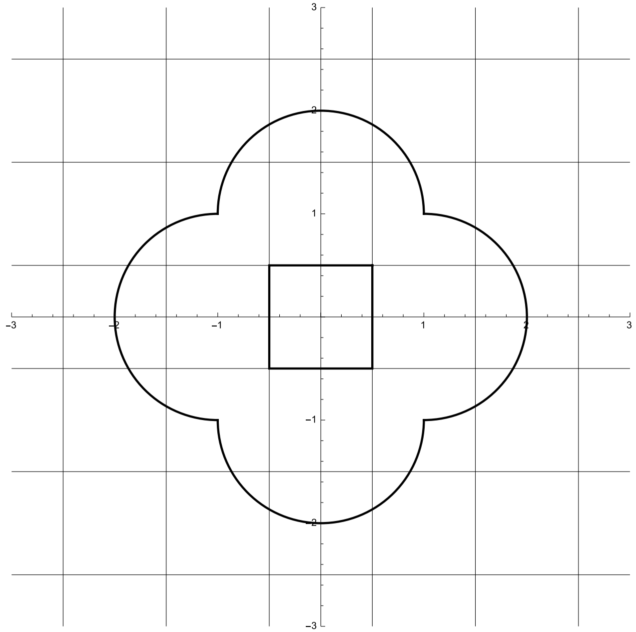
$$\mathfrak{o} = \left\{ n + m\sqrt{-1} : n, m \in \mathbb{Z} \right\}.$$

$$U = \left\{ z = x + yi \; : \; -\frac{1}{2} \leq x < \frac{1}{2}, -\frac{1}{2} \leq y < \frac{1}{2} \right\}$$

Define the Hurwitz continued fraction map $T : U \to U$ by

$$T(z) := \begin{cases} \frac{1}{z} - \left[\frac{1}{z}\right] & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases},$$

where $[w] = a \in \mathfrak{o}$ if $w \in a + U$.



$$a_n = a_n(z) \quad := \quad \begin{cases} \left[\dfrac{1}{T^{n-1}(z)}\right] & \text{if } T^{n-1}(z) \neq 0 \\ 0 & \text{if } T^{n-1}(z) = 0. \end{cases}$$

Then we get the continued fraction expansion of $z \in U$:

$$z = \frac{1}{|a_1(z)} + \frac{1}{|a_2(z)} + \cdots + \frac{1}{|a_n(z)} + \cdots .$$

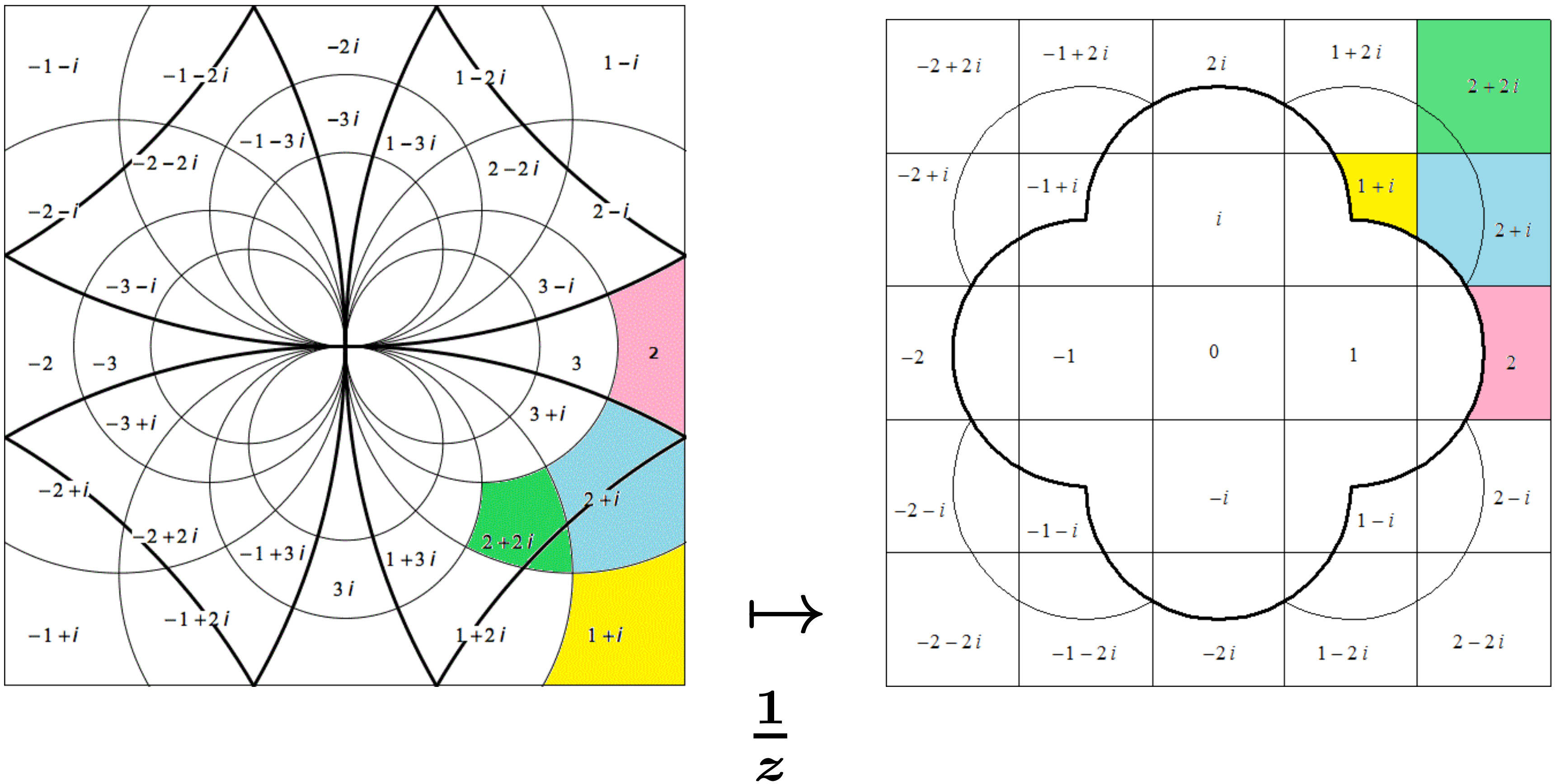


Fig. The cylinder sets $\langle a \rangle$ of U and $T\langle a \rangle$

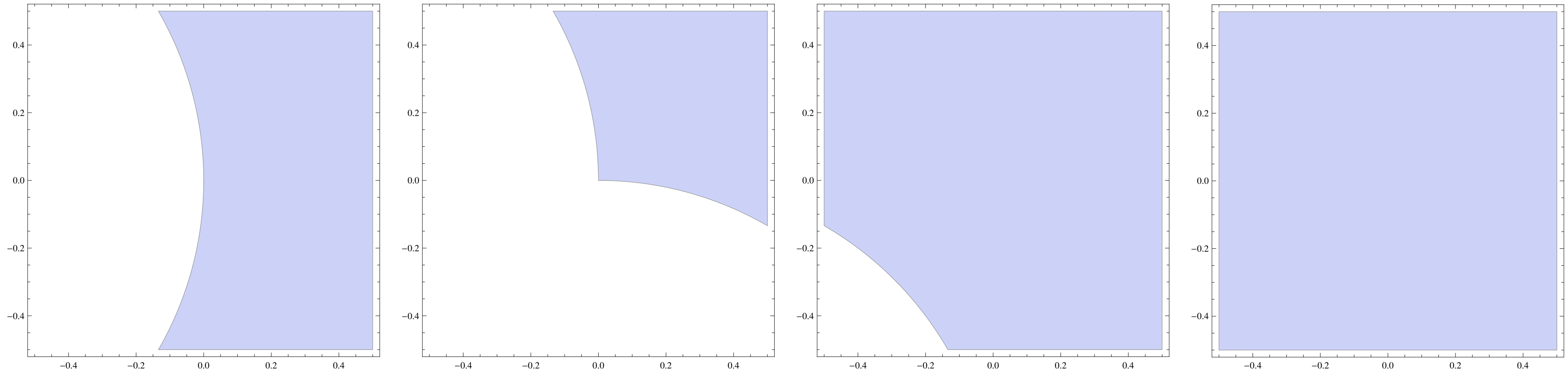


Fig.: $T\langle a\rangle$ for a cylinder set $\langle a\rangle$

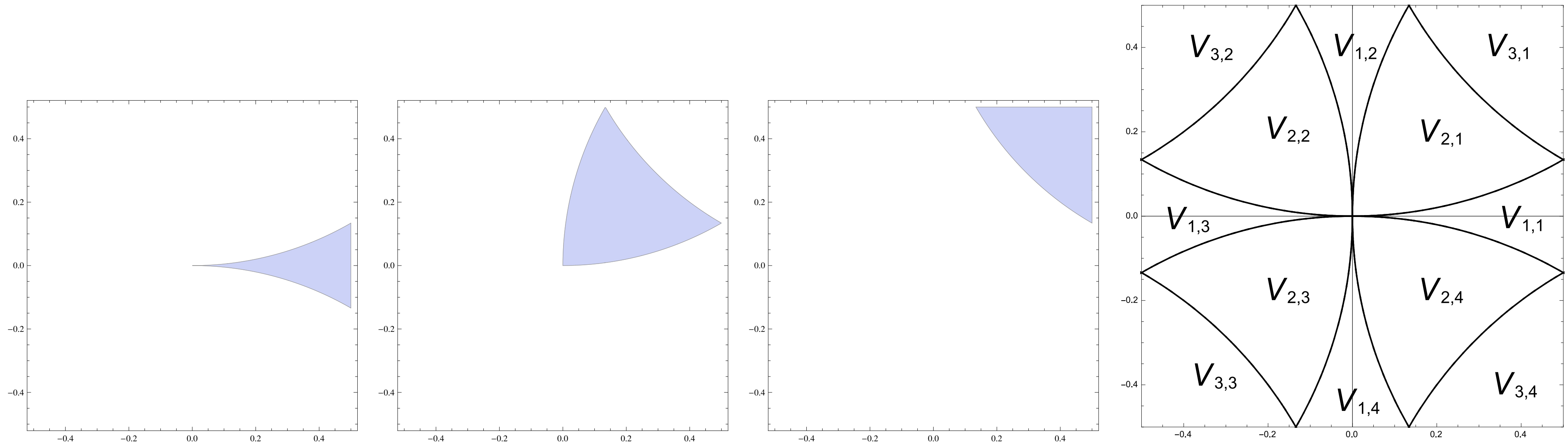
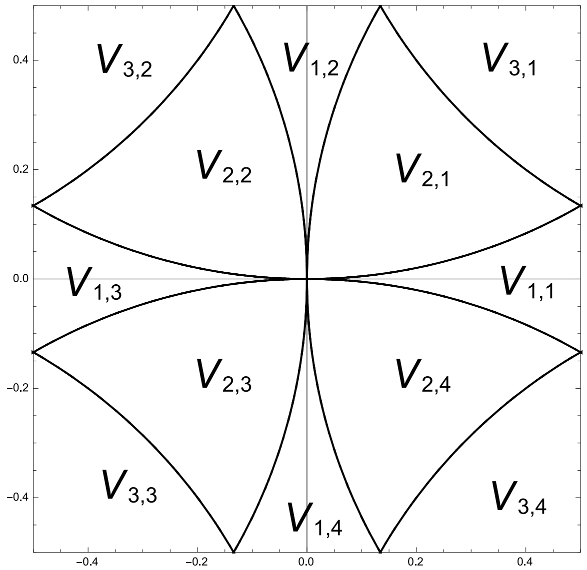


Fig. $V_{1,1}$, $V_{2,1}$, $V_{3,1}$ and the partition of U



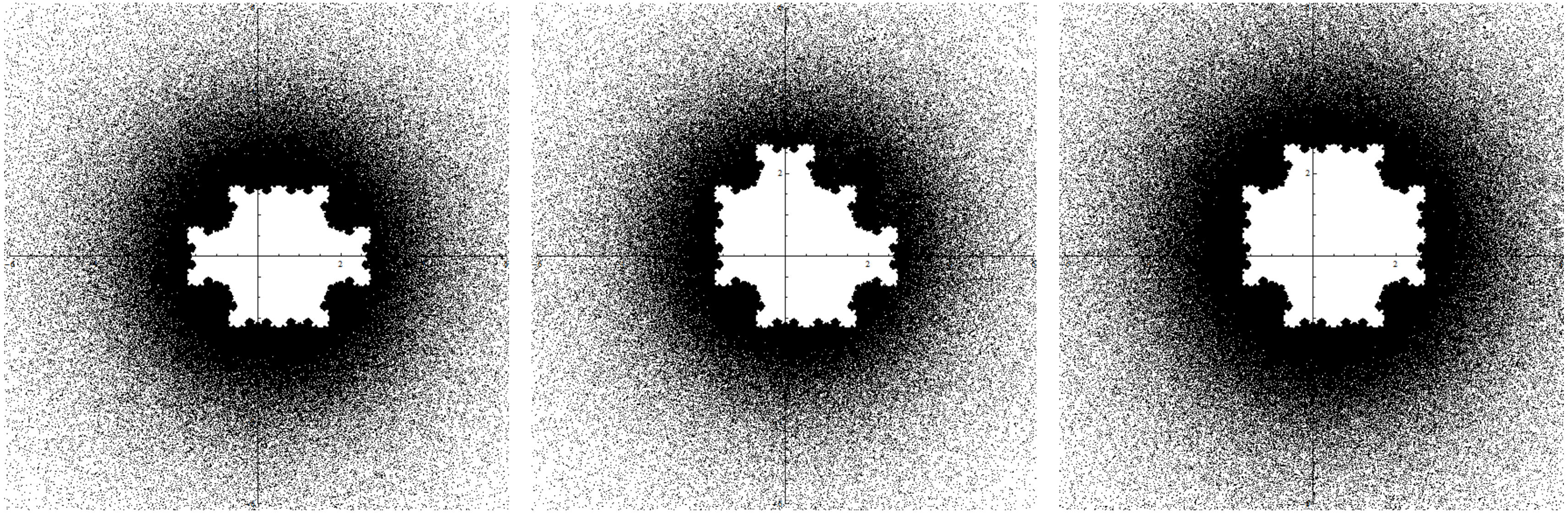
$$z = \frac{1}{|a_1} + \frac{1}{|a_2} + \cdots + \frac{1}{|a_n} \rightsquigarrow w = -\left(a_n + \frac{1}{|a_{n-1}} + \cdots + \frac{1}{|a_1}\right)$$

We define

$$V^*_{k,\ell} := \overline{\left\{-\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)}\right) : \begin{array}{l} z \in U, n \in \mathbb{N} \\ T^n(z) \in V_{k,\ell} \end{array}\right\}}$$

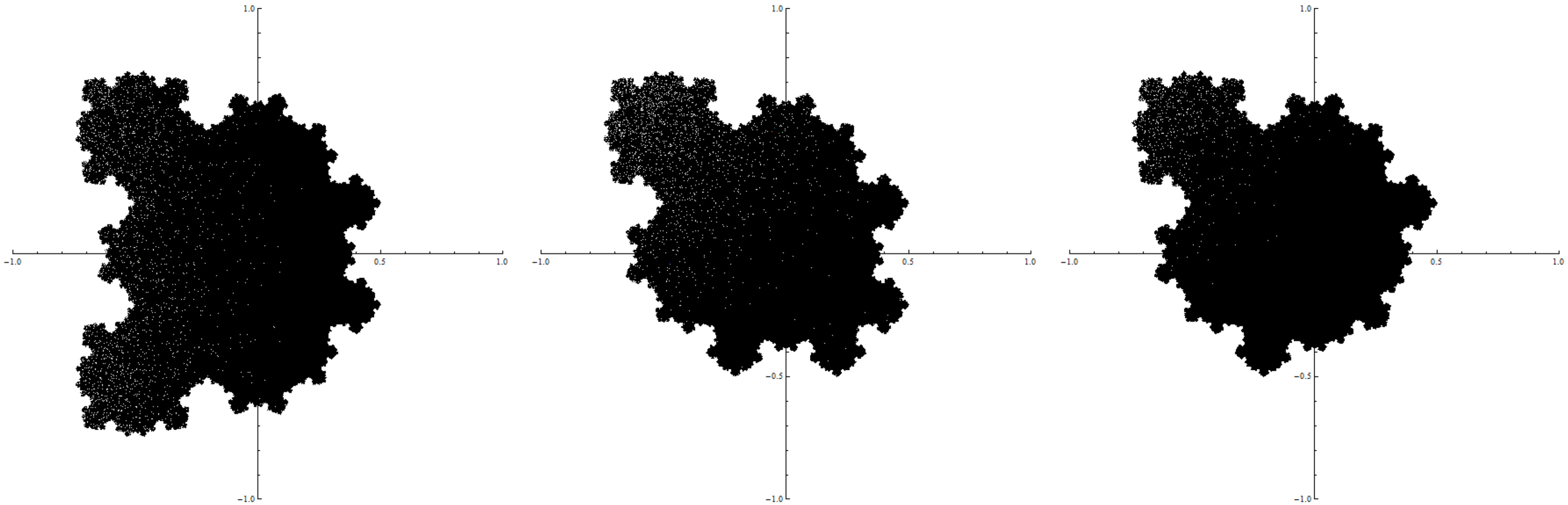
$$X_{k,\ell} := (V^*_{k,\ell})^{-1} = \left\{\frac{1}{w} : w \in V^*_{k,\ell}\right\}$$

for $1 \leq k \leq 3$ and $1 \leq \ell \leq 4$.



$V^*_{1,1}$ $V^*_{2,1}$ $V^*_{3,1}$

$\downarrow \frac{1}{z}$



$X_{1,1}$ $X_{2,1}$ $X_{3,1}$

Natural extension of T

We define

$$\hat{U} = \bigcup_{k=1}^{3} \bigcup_{\ell=1}^{4} V_{k,\ell} \times V_{k,\ell}^{*},$$

$$\hat{T}(z,w) = \left(\frac{1}{z} - \left[\frac{1}{z} \right], \frac{1}{w} - \left[\frac{1}{z} \right] \right)$$

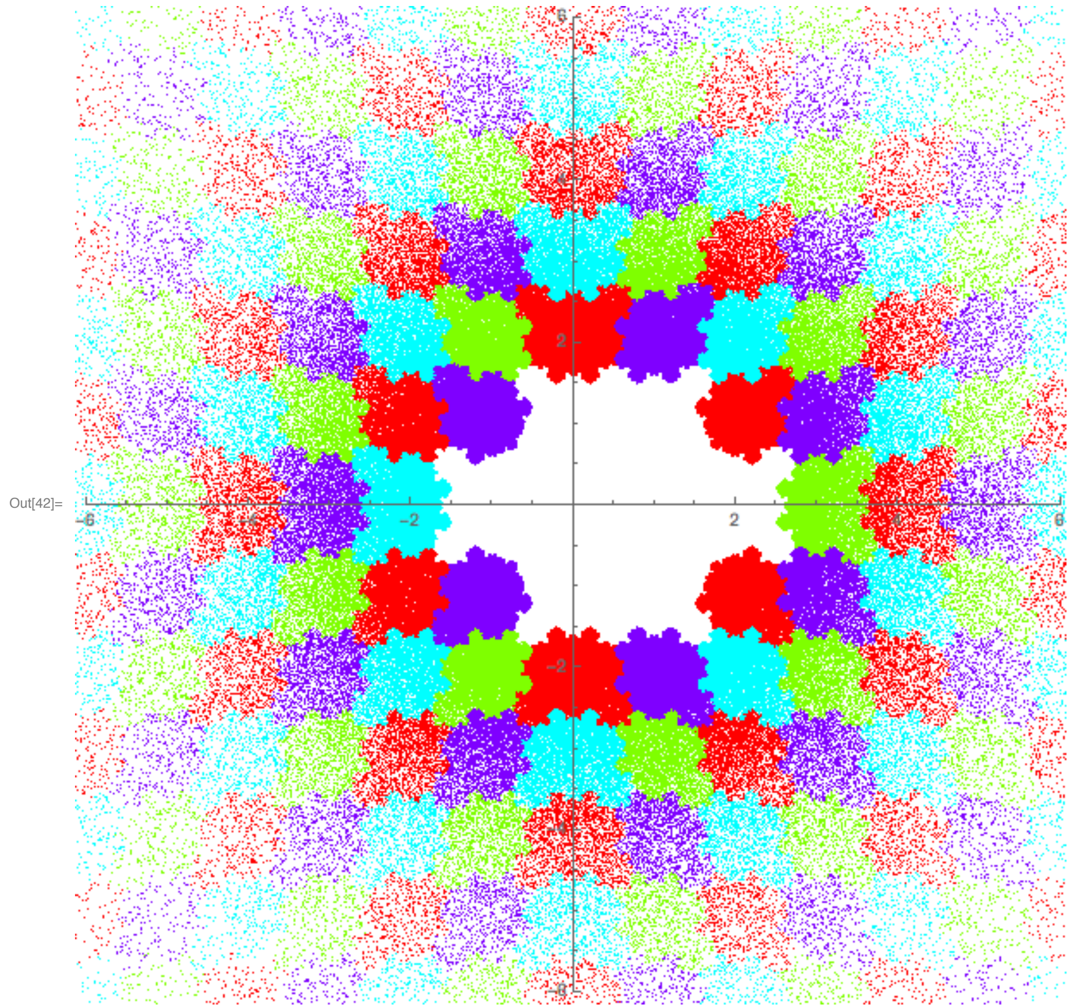
for $(z,w) \in \hat{U}$.

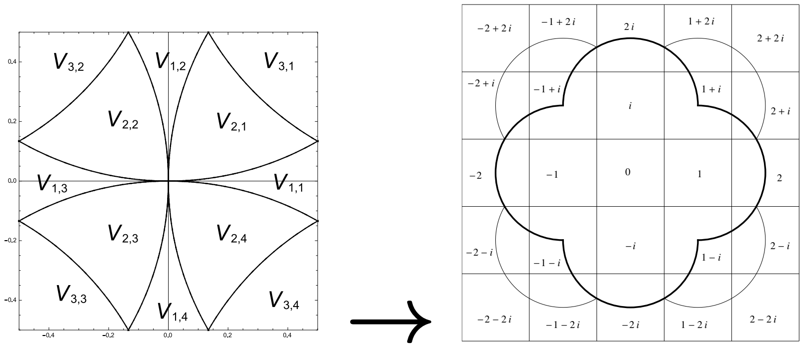
The density function of the invariant measure of T is

$$h(z) = \int_{V_{k,\ell}^{*}} \frac{1}{|z-w|^4} d\lambda(w)$$

for $z \in V_{k,\ell}$, where λ is the Lebesgue measure of $\mathbb{C}$.

Tilings





$X_{1,1}$:

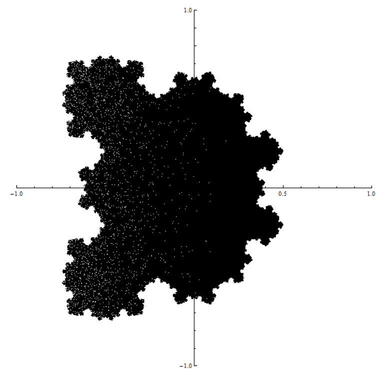


Fig.: $$V^*_{1,1} = \overline{\left\{ -\left(a_n(z) + \frac{1\,|}{|\,a_{n-1}(z)} + \cdots + \frac{1\,|}{|\,a_1(z)} \right) \; : \; \begin{array}{l} z \in U,\, n \in \mathbb{N} \\ T^n(z) \in V_{1,1} \end{array} \right\}}$$

Theorem (Ito-Nakada-Natsui-E)

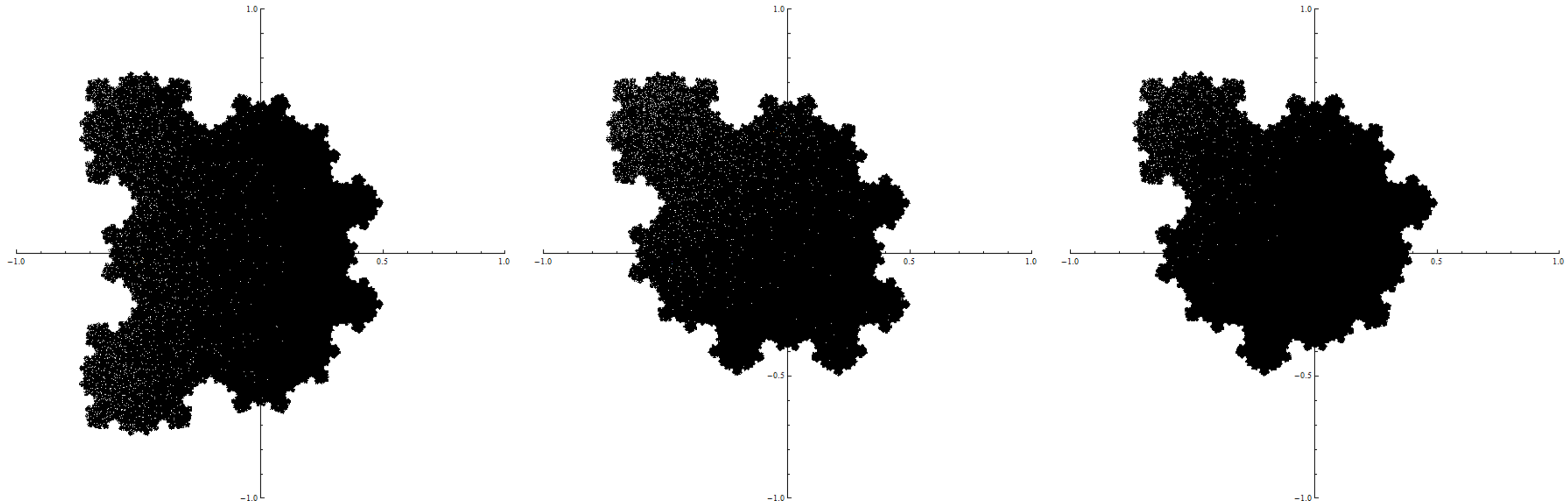
1. The origin point is an inner point of prototiles $X_{k,\ell}$ for any $1 \leq k \leq 3$, $1 \leq \ell \leq 4$.
2. The boundary of $X_{k,\ell} = (V^*_{k,\ell})^{-1}$ is a Jordan curve and has 2-dim Lebesgue measure 0 for any k, ℓ.
3. $V^*_{k,\ell}$ is tiled by the prototiles $\{X_{k,\ell}\}$ without overlapping for any k, ℓ. This means that the intersection of these tiles is a set of 2-dim Lebesgue measure 0.

Proposition (Ito-Nakada-Natsui-E)

1. The 1st and 2nd quadrants of $X_{1,1}$, $X_{2,1}$ and $X_{3,1}$ are coincide with each other.

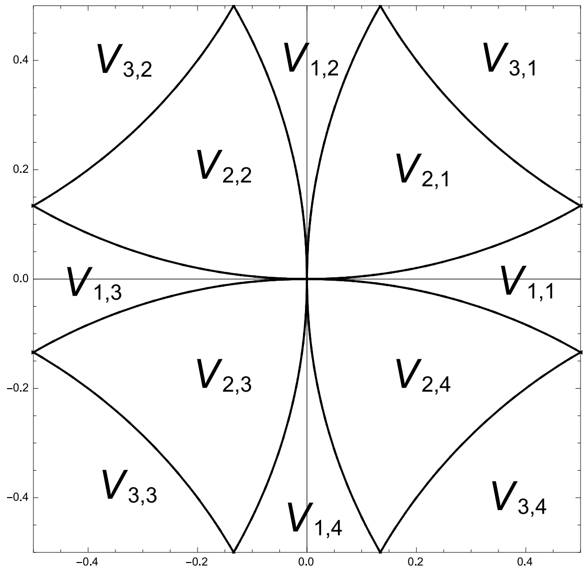
2. The 1nd, 2rd and 3th quadrants of $X_{2,1}$ and $X_{3,1}$ are coincide with each other.

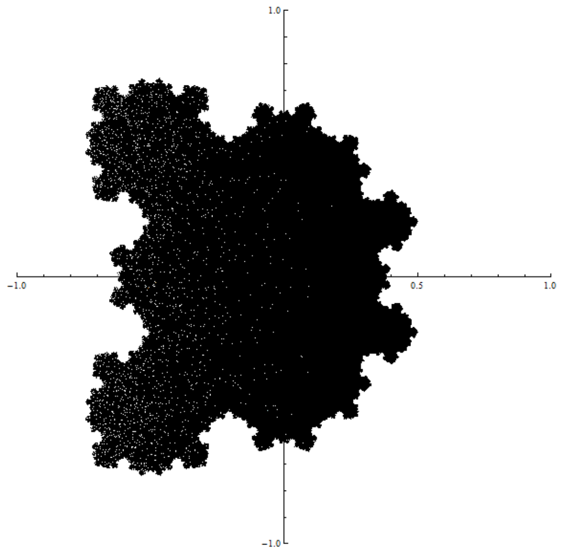
3. $\{z : |z| \leq 1\} \supset X_{1,1} \supset X_{2,1} \supset X_{3,1}$.

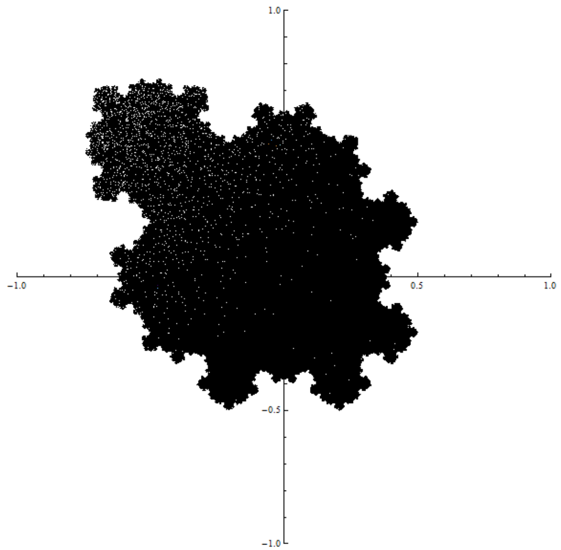


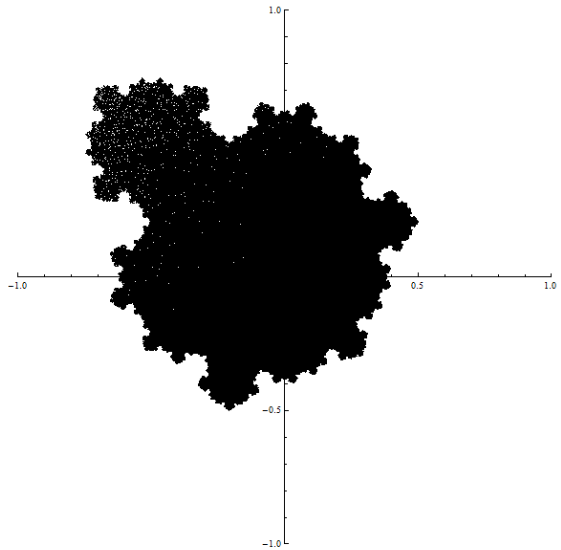
4. $X_{1,1}$ is x-axes symmetric.

5. $X_{2,1}$ and $X_{3,1}$ are symmetric to $y = -x$.



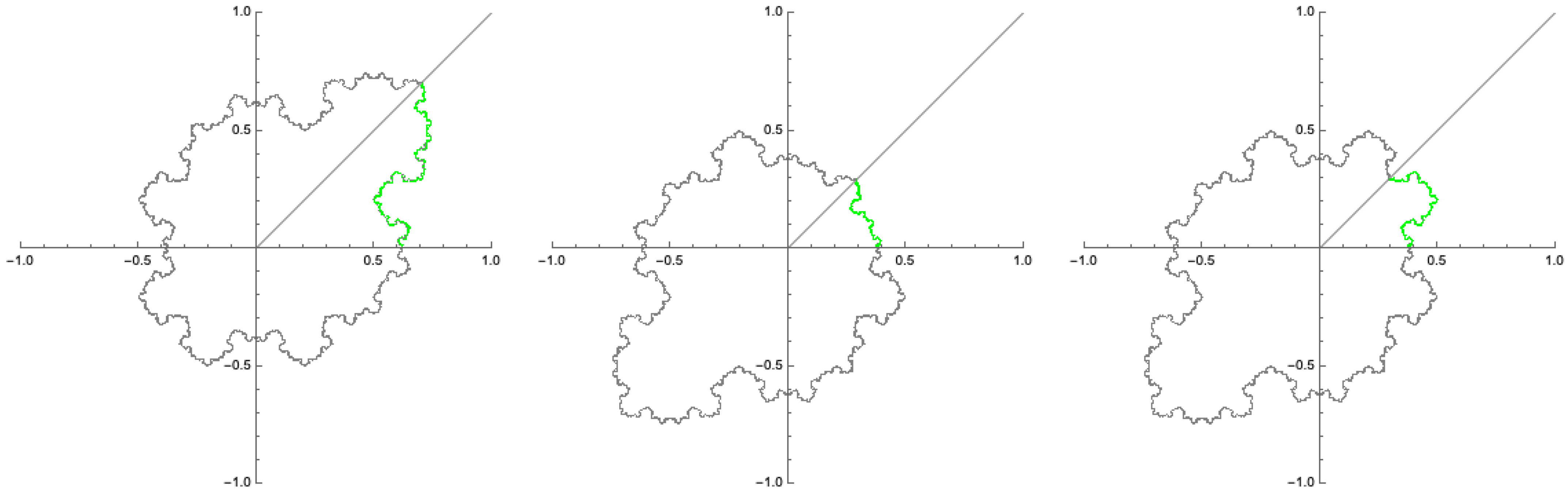




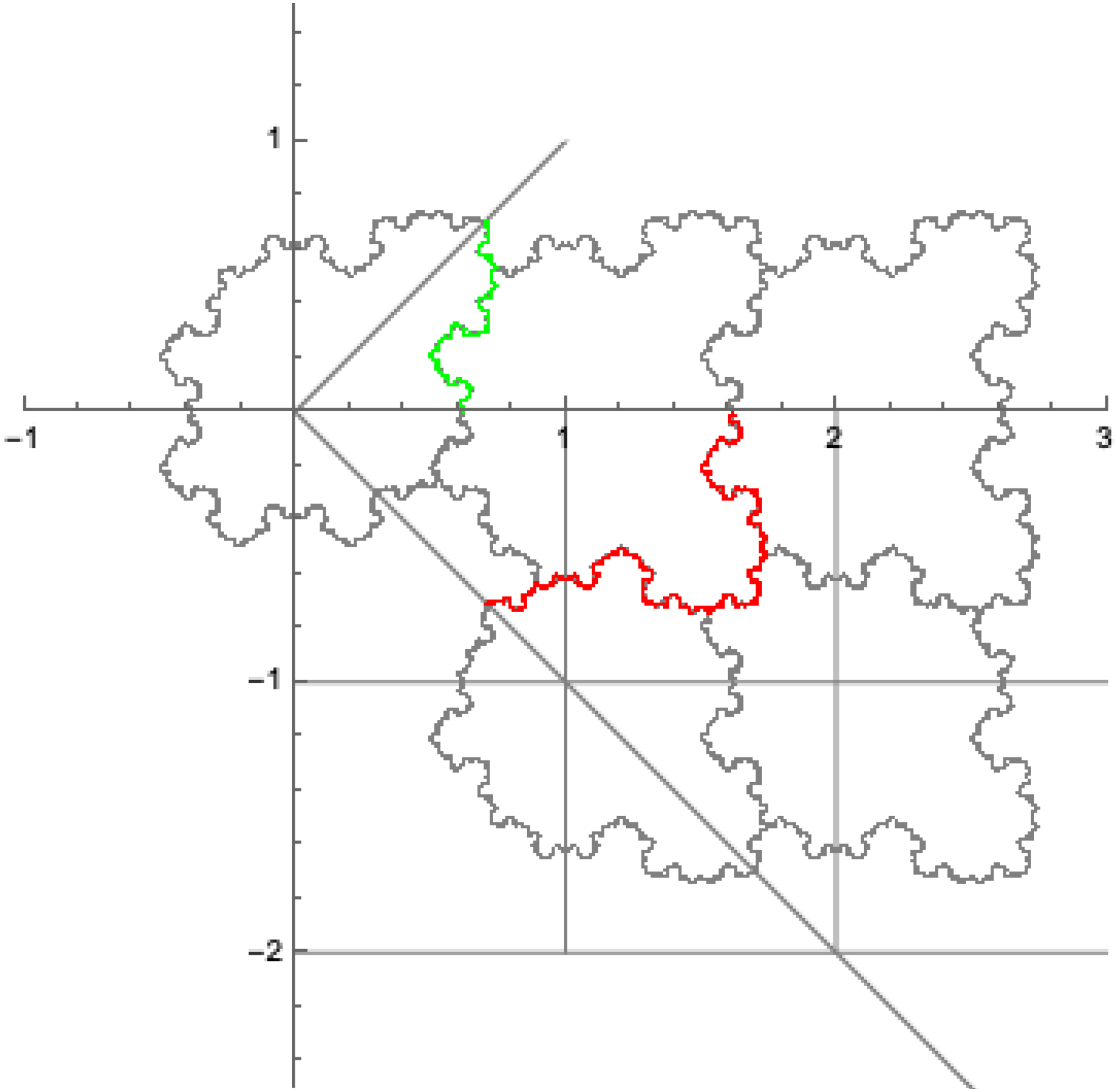


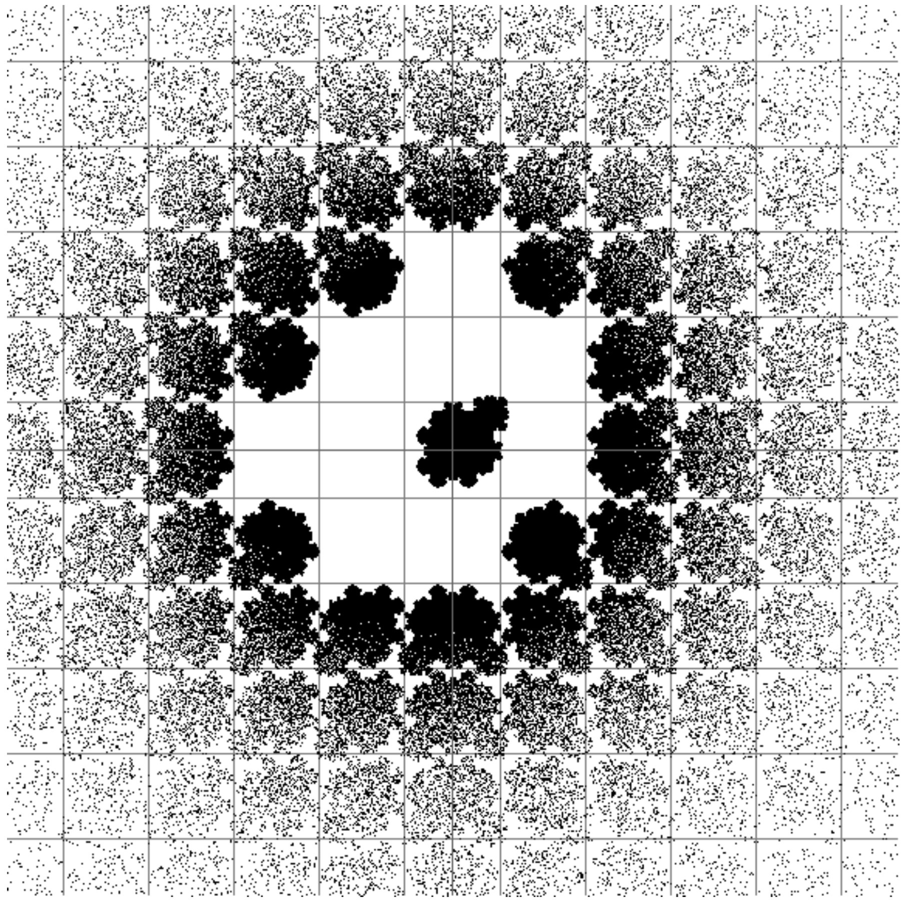
The boundaries of $X_{k,\ell}$

$$D := \{z \in \mathbb{C} : |z| \le 1, 0 \le \arg z \le \pi/4\}$$

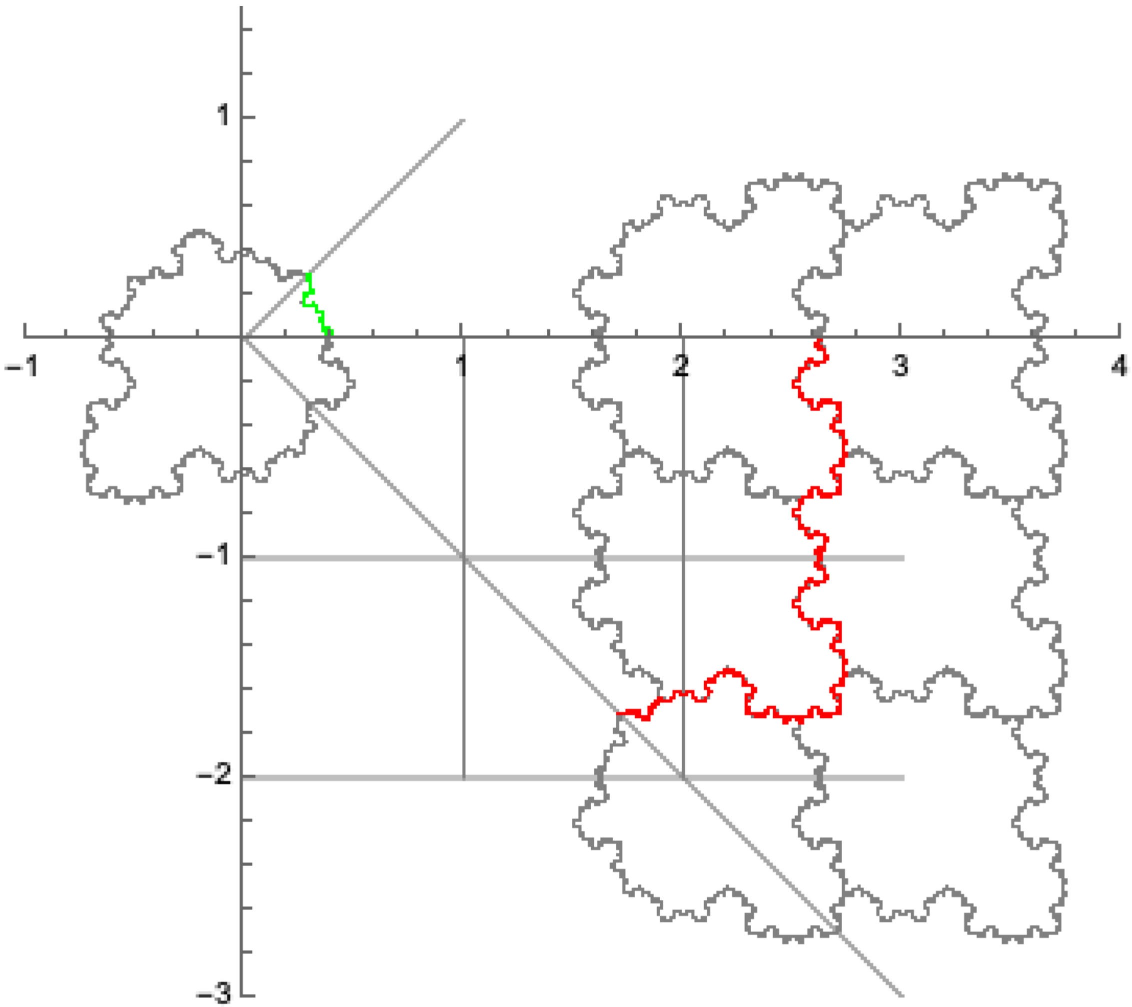


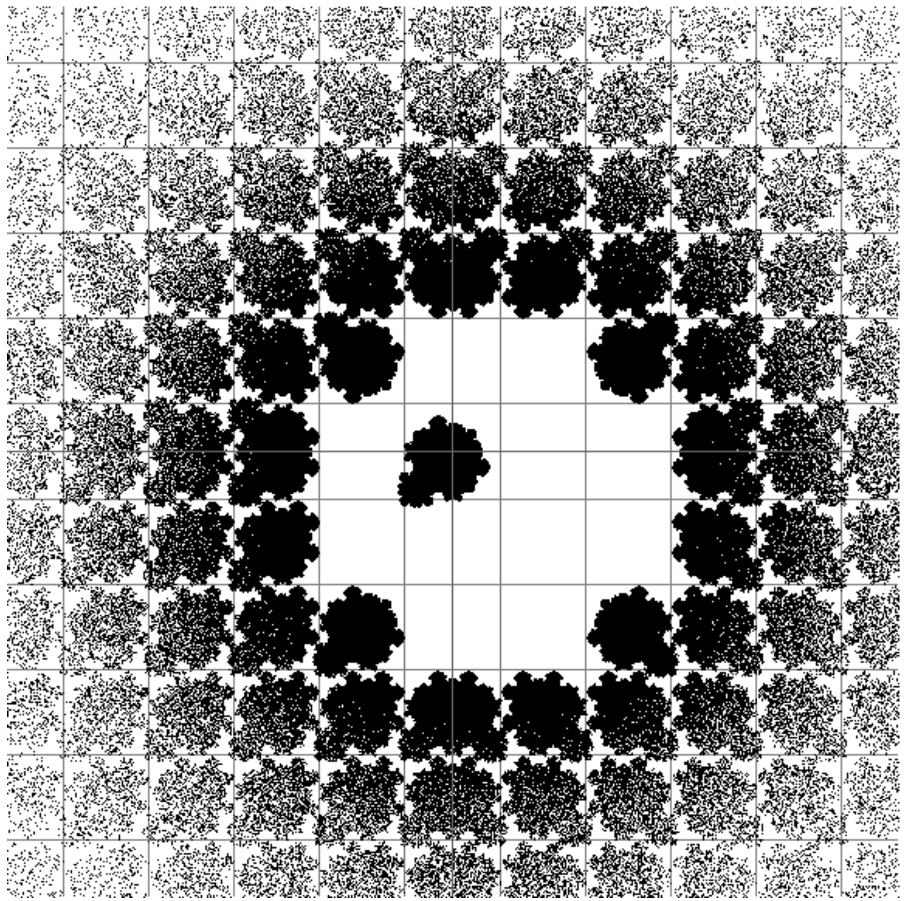
$$B_1 := \partial X_{2,1} \cap D, \ B_2 := \partial X_{3,4} \cap D, \ B_3 := \partial X_{2,4} \cap D.$$



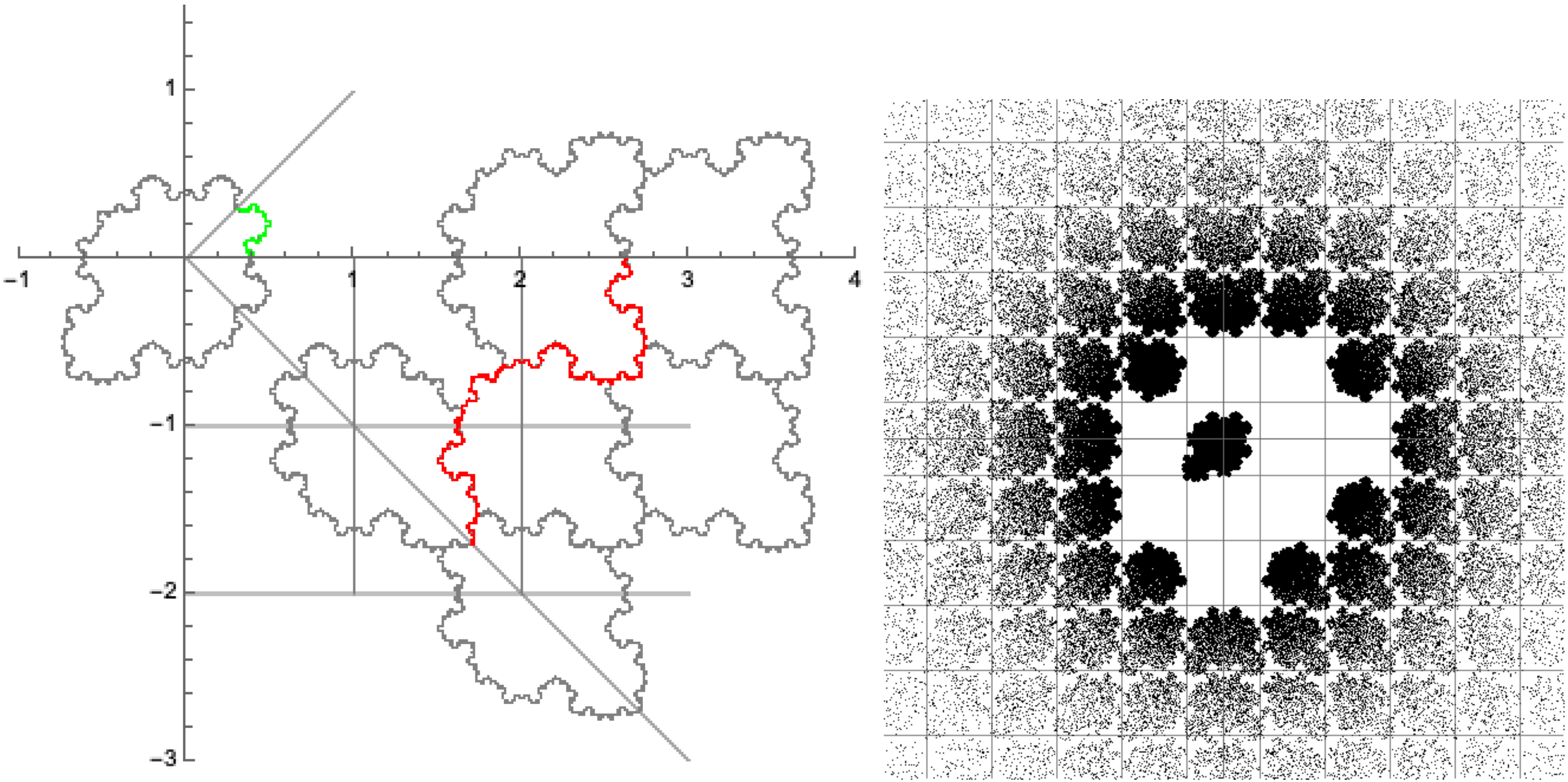


The green curve B_1 is mapped to the red curve by $f(z) = \frac{1}{z}$.





The green curve B_2 is mapped to the red curve by $f(z) = \frac{1}{z}$.



The green curve B_3 is mapped to the red curve by $f(z) = \frac{1}{z}$.

Define the following maps on $\mathbb{C}$:

$$
\begin{aligned}
&f_1^{(1)}(z) := (1 - i + iz)^{-1}\,,\ f_2^{(1)}(z) := (1 - iz)^{-1}\,,\\
&f_3^{(1)}(z) := (1 + \overline{z})^{-1}\,,\\
&f_1^{(2)}(z) := (2 - 2i + iz)^{-1}\,,\ f_2^{(2)}(z) := (2 - i - iz)^{-1}\,,\\
&f_3^{(2)}(z) := (2 - i + \overline{z})^{-1}\,, f_4^{(2)}(z) := (3 - i - \overline{z})^{-1}\,,\\
&f_5^{(2)}(z) := (2 + \overline{z})^{-1}\,,\\
&f_1^{(3)}(z) := (1 - i + \overline{z})^{-1}\,,\ f_2^{(3)}(z) := (2 - i - \overline{z})^{-1}\,,\\
&f_3^{(3)}(z) := (2 - i + iz)^{-1}\,, f_4^{(3)}(z) := (2 + iz)^{-1}\,,\\
&f_5^{(3)}(z) := f_5^{(2)}(z)
\end{aligned}
$$

All functions except for $f_1^{(1)}$ are contractions on $D \setminus B_\epsilon(0)$ for a small $\epsilon > 0$.

Theorem (E)

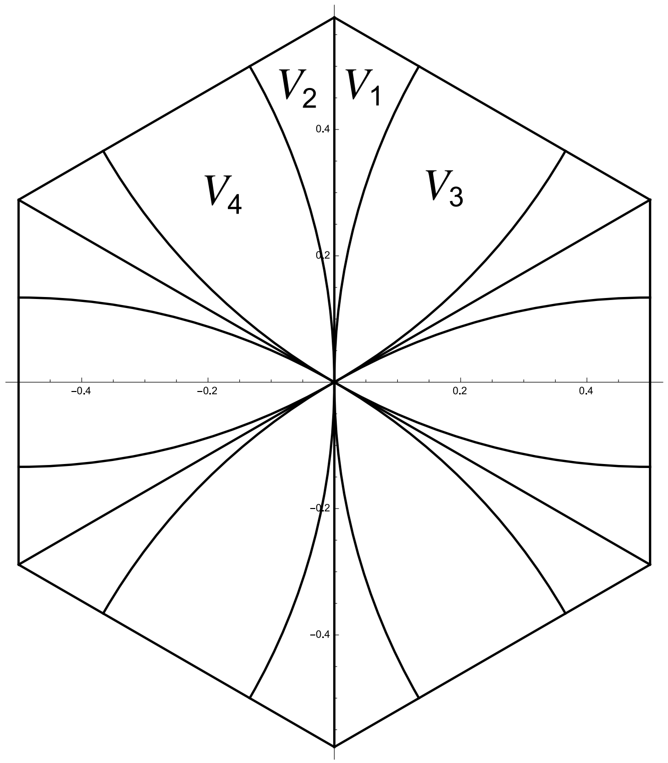
We have the following set equations:

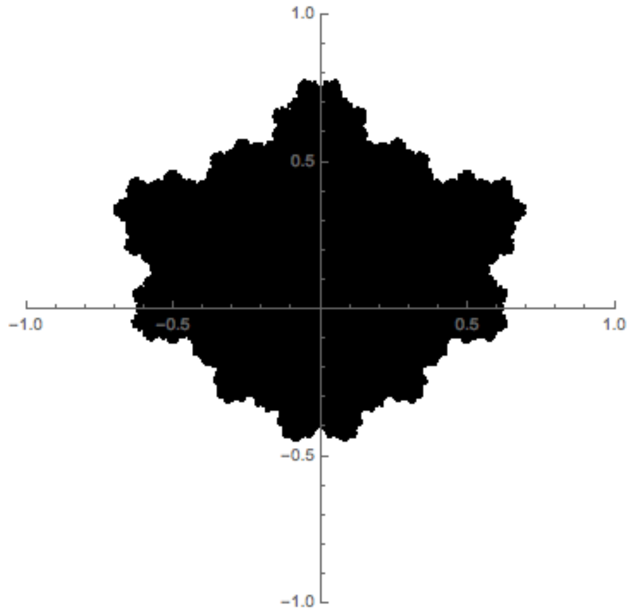
$$
\begin{aligned}
B_1 &= f_1^{(1)}(B_2) \cup f_2^{(1)}(B_1) \cup f_3^{(1)}(B_1) \\
B_2 &= f_1^{(2)}(B_2) \cup f_2^{(2)}(B_1) \cup f_3^{(2)}(B_1) \cup f_4^{(2)}(B_3) \cup f_5^{(2)}(B_1) \\
B_3 &= f_1^{(3)}(B_1) \cup f_2^{(3)}(B_2) \cup f_3^{(3)}(B_2) \cup f_4^{(3)}(B_1) \cup f_5^{(3)}(B_1)
\end{aligned}
$$

R. B. Lakein introduced eight different nearest integer complex continued fraction maps associated with $\mathbb{Q}(\sqrt{-d})$, $\boldsymbol{d = 1, 2, 3, 7, 11}$, including Hurwitz type.

H. Nakada, R. Natsui, and H.E
“On the ergodic theory of maps associated with the nearest integer complex continued fractions over imaginary quadratic fields”
Discrete Contin. Dyn. Syst. , no. 43 (11) (2023), 3883-3924.

The case of the nearest integer complex continued fraction map associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$.





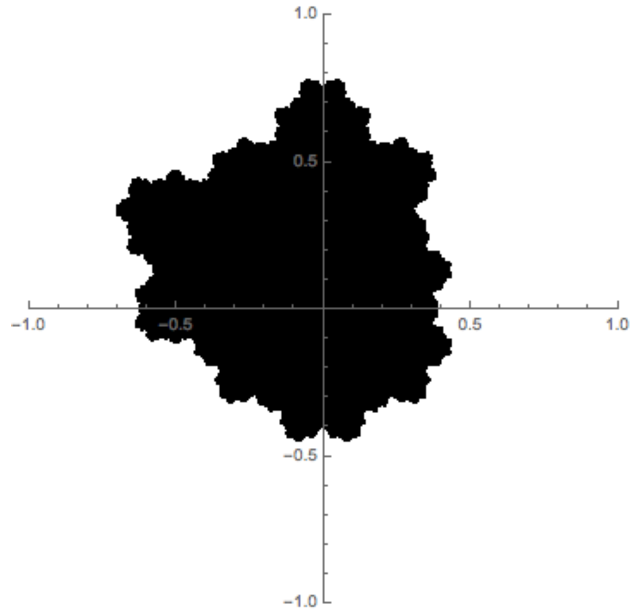


Fig.: The partition of U and the tiles X_1 and X_3

- **Can we do the same for these cases?**

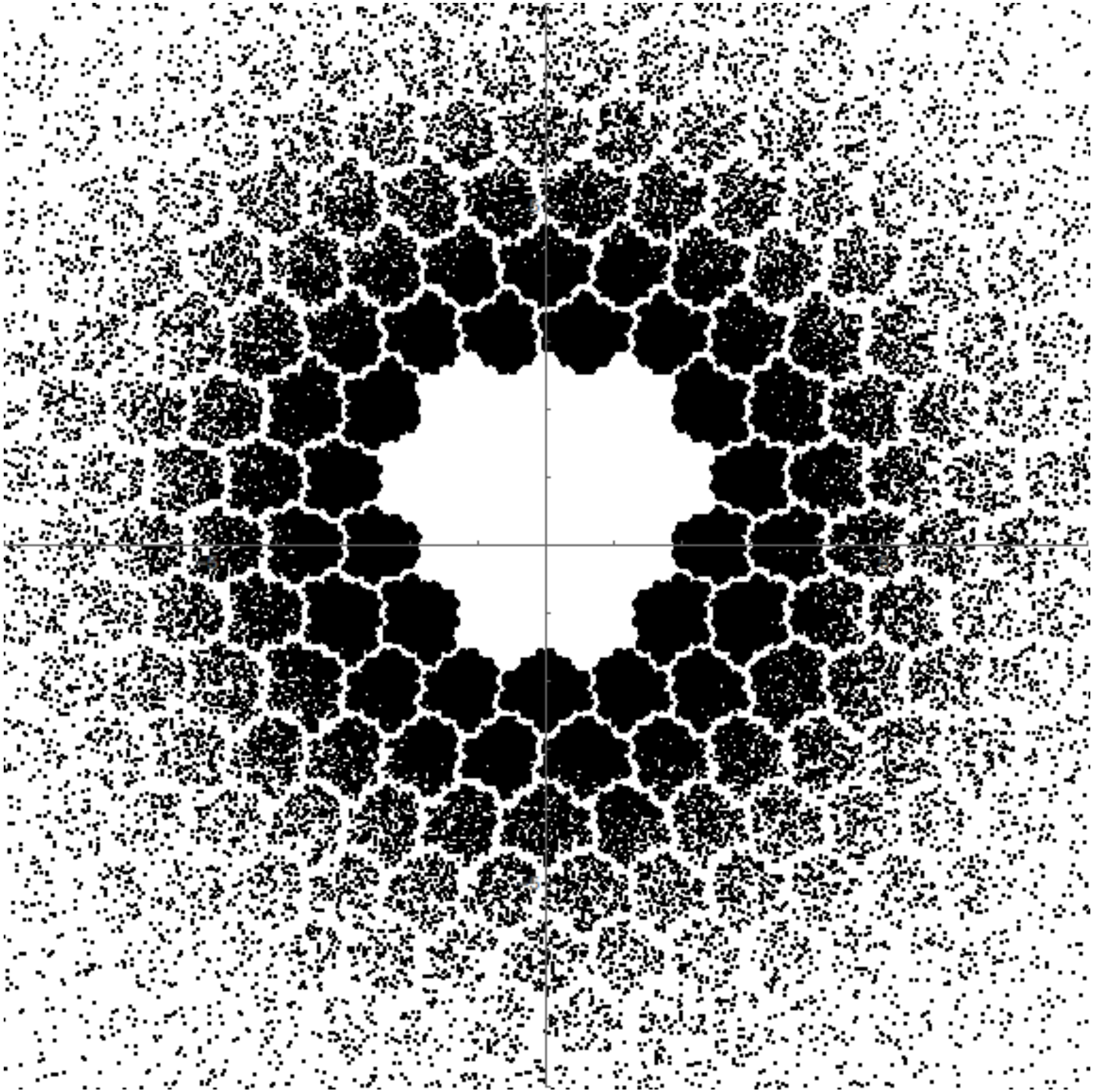


Fig: The tiling of V_1^* in the case of CF(3,H)

Thank you!