# Convergence and Combinatorics of the Reverse Algorithm

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# Definition

Let $\Lambda = \mathbb{P}^2$ and the subcones defined by

$$\Lambda_i = \{x \in \Lambda \mid 2x_i > x_1 + x_2 + x_3\},\ i \in \{1,2,3\},$$
$$\Lambda_4 = \Lambda \setminus (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3).$$

We define $M(x) = M_i$ if and only if $x \in \Lambda_i$, and then the map $f$ on $\Delta = \{x \in \Lambda \mid ||x||_1 = 1\}$ is given by

$$f(x) = \frac{M(x)^{-1}x}{||M(x)^{-1}x||_1}.$$

where

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ M_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

For the Reverse algorithm $(\Delta, f)$, we will use the set of digits $\mathcal{A} = \{1, 2, 3, 4\}$ corresponding to the partition $\Delta(1), \Delta(2), \Delta(3)$ and $\Delta(4)$, that is associated with $\Lambda_1, \Lambda_2, \Lambda_3$, and $\Lambda_4$.

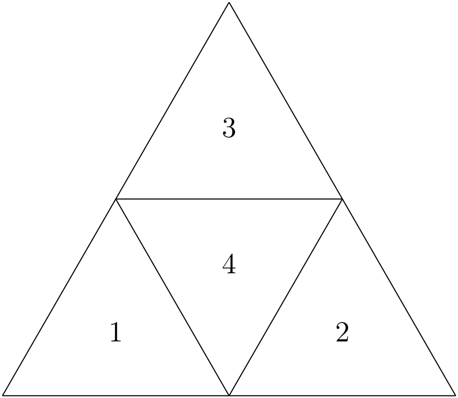


Figure 1: The partition of $\Delta$

Substitutions:

$$\sigma_1 : \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 21, \\ 3 \mapsto 31, \end{cases} \quad \sigma_2 : \begin{cases} 1 \mapsto 12, \\ 2 \mapsto 2, \\ 3 \mapsto 32, \end{cases} \quad \sigma_3 : \begin{cases} 1 \mapsto 13, \\ 2 \mapsto 23, \\ 3 \mapsto 3, \end{cases} \quad \sigma_4 : \begin{cases} 1 \mapsto 23, \\ 2 \mapsto 31, \\ 3 \mapsto 12. \end{cases}$$

$$\begin{aligned} \sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1(123) &= 121111113111111 \\ \sigma_1\sigma_2\sigma_3(111) &= 121312112131211213121 \\ \sigma_4\sigma_4\sigma_4(111) &= 122323311223233112232331 \end{aligned}$$

The natural extension and the explicit form of the invariant measure are known (Arnoux and Labbé, 2016). The density function is

$$h(x_0, x_1, x_2) = \frac{1}{(1-x_0)(1-x_1)(1-x_2)}.$$

Note that the invariant measure $\mu$ w.r.t. $h$ is finite.

# Theorems

**Theorem 1**

The Reverse algorithm $(\Delta, f)$ is ergodic with respect to $\mu$.

- We considered a **jump transformation** on $\Delta(4)$.
- We proved that **Rényi's condition** and **TC** are satisfied.

**Theorem 2**

The second Lyapunov exponent for the Reverse algorithm is negative.

- The second Lyapunov exponent is about $-0.103$. [Labbé, 2015]
- We considered separately the cases with and without the singular point. [Berthé et al., 2021]

**Theorem 3**

If $\boldsymbol{\tau} \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}^{\mathbb{N}}$ is a directive sequence in which the block $(\sigma_1, \sigma_2, \sigma_3)^7$ occurs with bounded gaps, then $\mathcal{L}_{\boldsymbol{\tau}}$ is balanced.

# On the proof of Theorem 3

## Lemma 1

Let $u \in \mathcal{A}^*$ be given. If $u$ is $C$-balanced then $\sigma_i(u)$ is $(C+4)$-balanced for $i \in \{1,2,3\}$ and $(C+2)$-balanced for $i = 4$.

## Lemma 2

For each $m \leq n$, we have $\|(M_{\tau_{[m,n)}})|_{\mathbf{1}_n^\perp}\|_\infty \leq 8$.

$M_{\tau_{[m,n)}}(\ell(u)) = \ell(\tau_{[m,n)}(u)), \ \ \tau_i \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \}$

## Lemma 3 [Delecroix et al., 2013]

If $M_{\tau_{[n,n+3)}} = M_{\sigma_1} M_{\sigma_2} M_{\sigma_3}$, we have

$$\|(M_{\tau_{[n,n+3)}})|_{\mathbf{1}_{n+3}^\perp}\|_\infty \leq \frac{5}{7}.$$

$8 \cdot (5/7)^7 < 1$

# References

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Thank you for your attention!