The Higher order partial derivatives of Okamoto's functions with respect to the parameter

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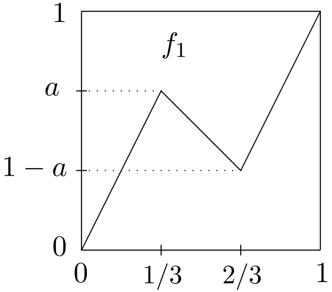
Motivation/Background

Okamoto's self-affine functions (2005)

Okamoto introduced a one-parameter family of self-affine functions $\{F_a; 0 < a < 1\}$ on the interval $[0, 1]$.

Construction: Fix a parameter $0 < a < 1$.



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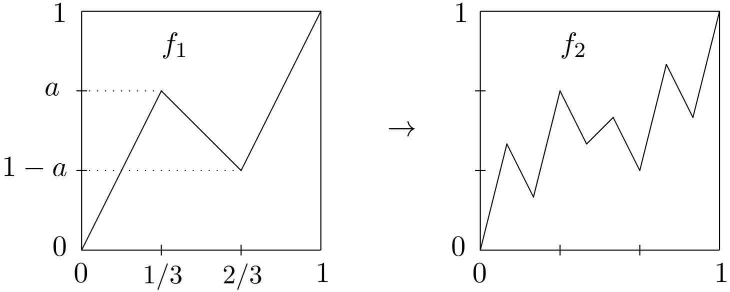

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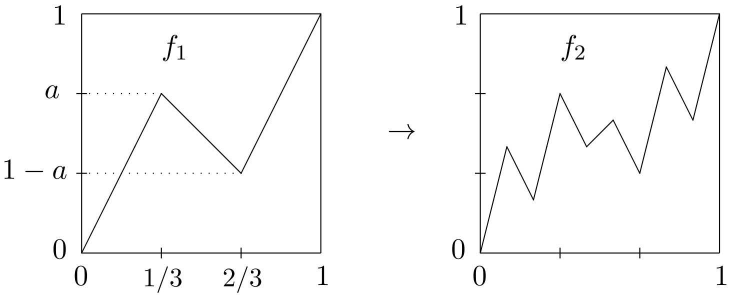

- Continuing this way, obtain a sequence of continuous functions $f_n : [0,1] \to [0,1]$.
- Let $F_a(x) := \lim_{n\to\infty} f_n(x)$.

Examples of the graph of $F_a(x)$

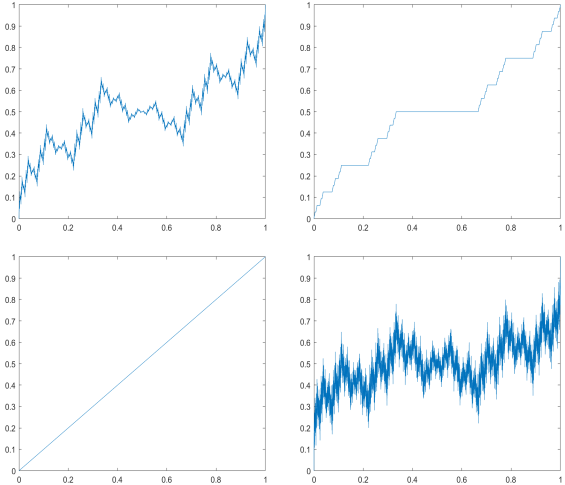

Figure: $F_{\frac{2}{3}}(x)$ (Bourbaki's Function), $F_{\frac{1}{2}}(x)$ (Cantor's Devil's staircase), $F_{\frac{1}{3}}(x)$ (Identity Function), $F_{\frac{5}{6}}(x)$ (Perkins's Function)

Some remarks

- F_a is continuous, mapping $[0,1]$ onto itself
- F_a is self-affine
- F_a is singular for $a \leq 1/2, a \neq 1/3$
- F_a is of unbounded variation when $a > 1/2$

Differentiability of Okamoto's self-affine functions

Theorem (Okamoto 2005, Kobayashi 2009)

Let $a_0 \approx .5592$ *be the unique root in* $(0, 1)$ *of*

$$54a^3 - 27a^2 = 1.$$

1. *If* $2/3 \leq a < 1$, *then* F_a *is nowhere differentiable.*
2. *If* $a_0 \leq a < 2/3$, *then* F_a *is non-differentiable almost everywhere, but differentiable at uncountably many points.*
3. *If* $0 < a < a_0$ *and* $a \neq 1/3$, *then* F_a *is differentiable almost everywhere, but non-differentiable at uncountably many points.*
4. *If* $a = 1/3$, F_a *is differentiable everywhere.*

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Theorem (Allaart, 2016)

The Hausdorff dimension of the exceptional sets in (ii) and (iii) can be computed explicitly.

Self-affinity plays a rule!

Okamoto's Function: $F_a(x)$

$$F_a(x) = \begin{cases} aF_a(3x), & 0 \le x \le \frac{1}{3} \\ (1-2a)F_a(3x-1) + a, & \frac{1}{3} \le x \le \frac{2}{3} \\ aF_a(3x-2) + (1-a), & \frac{2}{3} \le x \le 1 \end{cases}$$

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Theorem (McCollum, 2011)

$$\dim_B \mathrm{Graph}(F_a) = \begin{cases} 1 & \textit{if } 0 < a \le 1/2, \\ 1 + \log_3(4a-1) & \textit{if } 1/2 < a < 1. \end{cases}$$

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Theorem (Barany-Prokaj, 2025)

$$\dim_H \mathrm{Graph}(F_a) = \dim_B \mathrm{Graph}(F_a)$$

for Lebesgue-almost all $a \in (1/2, 1)$.

Who is Okamoto?

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Figure: Dr. Hisashi Okamoto

Questions

First partial derivative of F_a w.r.t $a = 1/3$

Remark

- $F_{1/3}(x) = x$ so that $\frac{d}{dx}F_{1/3}(x) = 1$.
- $F_a(x)$ is an analytic function w.r.t. $a \in (0, 1)$.

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- $F_a(x)$ is an analytic function w.r.t. $a \in (0, 1)$.

Define the first partial derivative of $F_a(x)$ with respect to $a = 1/3$:

$$M_{1,1/3}(x) := \left. \frac{\partial F_a(x)}{\partial a} \right|_{a=1/3}, \qquad x \in [0, 1].$$

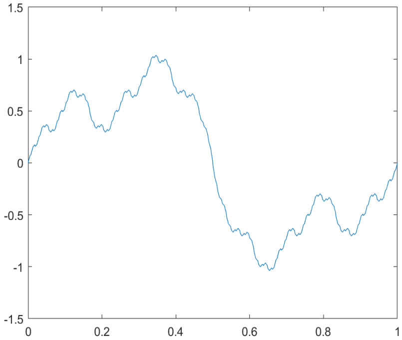
First Question

1. What does the graph of $M_{1,1/3}(x)$ look like?
2. At which points x is $M_{1,1/3}$ differentiable?

Result from Undergraduate Research group

Theorem (Dalaklis-K.-Mathis-Paizanis, 2021)

$M_{1,1/3}(x) := \left.\frac{\partial F_a(x)}{\partial a}\right|_{a=1/3}$ is continuous, but it does not possess a finite derivative at any point on $[0,1]$.



More questions come out!

Define the k-th partial derivative of $F_a(x)$ with respect to $a \in (0,1)$:

$$M_{k,a}(x) := \frac{\partial^k F_a(x)}{\partial a^k}, \qquad k = 1, 2, 3, \cdots$$

Next Question

What does the graph of $M_{k,a}(x)$ look like?

Graphs of $M_{k,a} := \frac{\partial^k F_a(x)}{\partial a^k}$ for various a and k

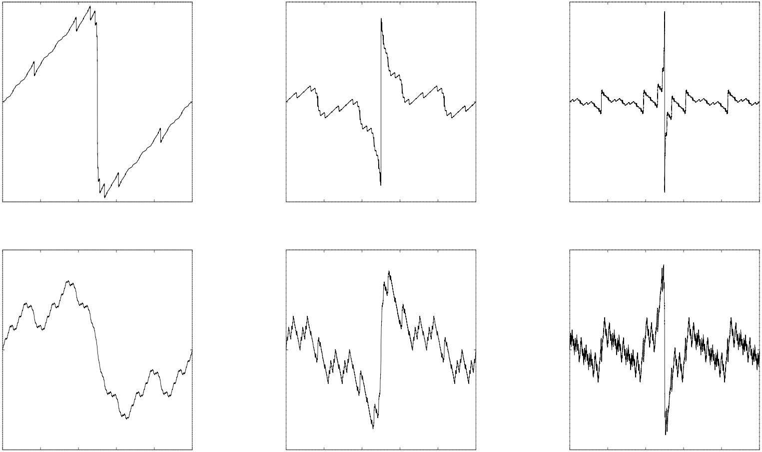


Figure: Top row: $a = 1/6$; second row: $a = 1/3$. In each row, from left to right, $k = 1$, $k = 2$ and $k = 3$.

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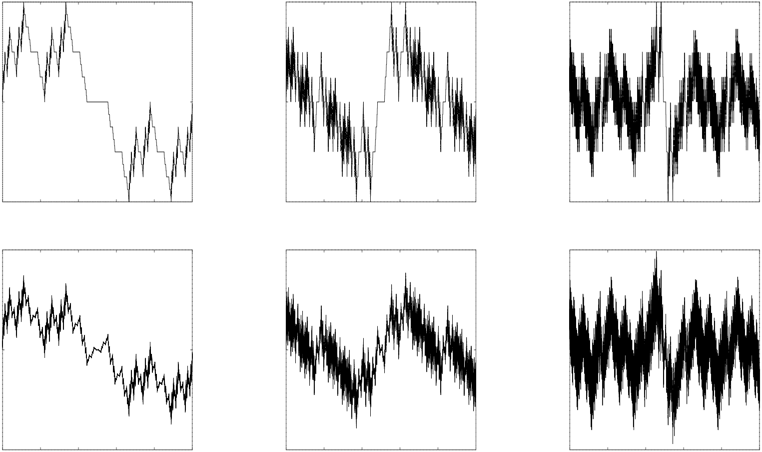

Figure: Top row: $a = 1/2$; second row: $a = a_0 \approx .5592$. In each row, from left to right, $k = 1$, $k = 2$ and $k = 3$.

Observation from computer graphs

For each $k \in \mathbb{N}$ and $a \in (0,1)$,

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Question

1. What is the box-counting dimension of $\mathrm{Graph}(M_{k,a})$?
2. What is the differentiability of $M_{k,a}(x)$?

Box-counting dimension of $\mathrm{Graph}(M_{k,a})$

Recall that F_a is strictly self-affine; however, $M_{k,a}$ is NOT strictly self-affine!

$$M_{1,a}(x) = \begin{cases} aM_{1,a}(3x) + F_a(3x), & 0 \le x \le \frac{1}{3} \\ (1-2a)M_{1,a}(3x-1) - 2F_a(3x-1) + 1, & \frac{1}{3} \le x \le \frac{2}{3} \\ aM_{1,a}(3x-2) + F_a(3x-2) - 1, & \frac{2}{3} \le x \le 1. \end{cases}$$

For $k \ge 2$,

$$M_{k,a}(x) = \begin{cases} aM_{k,a}(3x) + kM_{k-1,a}(3x), & 0 \le x \le \frac{1}{3} \\ (1-2a)M_{k,a}(3x-1) - 2kM_{k-1,a}(3x-1), & \frac{1}{3} \le x \le \frac{2}{3} \\ aM_{k,a}(3x) + kM_{k-1,a}(3x-2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Proposition (Symmetry)

For each $k \in \mathbb{N}$, $M_{k,a}(1-x) = -M_{k,a}(x), \qquad x \in [0,1]$.

Results

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Theorem (Allaart-Dalaklis-K.-Ortiz-Zheng 2025)

For each $k \in \mathbb{N}$ *and* $a \in (0,1)$, *we have*

$$\begin{aligned} \dim_B \mathrm{Graph}(M_{k,a}) &= \dim_B \mathrm{Graph}(F_a) \\ &= \begin{cases} 1 + \log_3(4a-1) & \text{if } a \geq 1/2, \\ 1 & \text{if } a \leq 1/2. \end{cases} \end{aligned}$$

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Note: $\dim_B \mathrm{Graph}(M_{k,a})$ does NOT change as k increases!

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- It's nearly the same three cases as in Okamoto's theorem.
- This result does not show any influence from k.

Question

1. What are the possible values of $M'_{k,a}(x) = \frac{d}{dx}M_{k,a}(x)$?
2. At which points x is $M_{k,a}$ differentiable? How large is this set for each k?

Main Questions

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Proposition

For each $x \in (0,1)$, $M'_{k,a}(x) = 0$ or $\pm\infty$, or does not exist.

Some notation

- Ternary expansion of $x \in [0,1)$:

$$x = \sum_{i=1}^{\infty} x_i 3^{-i}, \qquad x_i \in \{0,1,2\}.$$

(For $x = k/3^i$ with $k \in \mathbb{Z}$, take the expansion ending in 0^∞.)

- Number of 1's in the first n ternary digits of x:

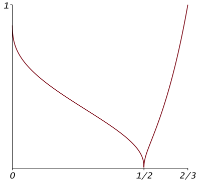
$$l_n(x) := \#\{i \le n : x_i = 1\}.$$

Definitions

Define

$$\phi(a) := \frac{\log(3a)}{\log a - \log|2a-1|}, \qquad a \in (0, 2/3]\backslash\{1/3, 1/2\}.$$

Set $\phi(0) := 1, \phi(1/3) := 1/3$, and $\phi(1/2) := 0$.



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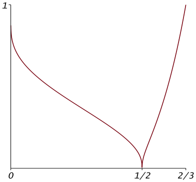

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Note: ϕ is strictly decreasing on $[0, 1/2]$ and strictly increasing on $[1/2, 2/3]$.

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Note:

- $C_0 < 0$ for $a < 1/3$, whereas $C_0 > 0$ for $a > 1/3$.
- $C_0(a)$ tends to $-\infty$ as $a \to (1/3)-$, to $+\infty$ as $a \to (1/3)+$, and to 0 as $a \to 1/2$.

Main Results

Theorem (Allaart-Dalaklis-K.-Ortiz-Zheng 2025)

Let $k \geq 1$ *and* $a \in (0,1)$. *For* $x \in (0,1)$ *and* $n \in \mathbb{N}$, *define*

$$r_n(x) := l_n(x) - n\phi(a).$$

(a) *If* $1/3 < a < 2/3$ *and* $a \neq 1/2$, *then* $M'_{k,a}(x) = 0$ *iff*

$$r_n(x) - kC_0 \log n \to \infty \qquad \text{as } n \to \infty;$$

(b) *If* $0 < a < 1/3$, *then* $M'_{k,a}(x) = 0$ *iff*

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Proposition

If $a = 1/2$, *then* $M'_{k,a}(x) = 0$ *iff there exists an* $n \in \mathbb{N}$ *such that* $l_n(x) = k+1$ *and* $3^n x \notin \mathbb{Z}$.

My favorite Corollary!

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$$\mathcal{D}^0_{0,a} \supsetneq \mathcal{D}^0_{1,a} \supsetneq \mathcal{D}^0_{2,a} \supsetneq \cdots.$$

Moreover, for each $k \geq 0$, $\mathcal{D}^0_{k,a} \setminus \mathcal{D}^0_{k+1,a}$ *has positive Hausdorff dimension.*

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Yes! $M_{k,a}(x)$ becomes progressively "less differentiable" as k increases!

Thank YOU for listening!