Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

Guangdong University of Technology
Yao-Qiang Li
yaoqiang.li@gdut.edu.cn

Numeration and Substitution 2025
University of Tsukuba
12/09/2025

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1.

2.

3.

4.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

2.

3.

4.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

2. Supports of infinite convolutions

3.

4.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

2. Supports of infinite convolutions

3. Infinite sums of union sets

4.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

2. Supports of infinite convolutions

3. Infinite sums of union sets

4. Spectral measures with and without compact supports of arbitrary dimensions

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background
2. Supports of infinite convolutions
3. Infinite sums of union sets
4. Spectral measures with and without compact supports of arbitrary dimensions

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background   1.2 Main results
2. Supports of infinite convolutions
3. Infinite sums of union sets
4. Spectral measures with and without compact supports of arbitrary dimensions

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background   1.2 Main results
2. Supports of infinite convolutions
   2.1 Background   2.2 Main results
3. Infinite sums of union sets
4. Spectral measures with and without compact supports of arbitrary dimensions

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

   1.1 Background 1.2 Main results

2. Supports of infinite convolutions

   2.1 Background 2.2 Main results

3. Infinite sums of union sets

   3.1 Background 3.2 Main results

4. Spectral measures with and without compact supports of arbitrary dimensions

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background   1.2 Main results
2. Supports of infinite convolutions
   2.1 Background   2.2 Main results
3. Infinite sums of union sets
   3.1 Background   3.2 Main results
4. Spectral measures with and without compact supports of arbitrary dimensions
   4.1 Background   4.2 Main results

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

1.1 Background 1.2 Main results

2. Supports of infinite convolutions

2.1 Background 2.2 Main results

3. Infinite sums of union sets

3.1 Background 3.2 Main results

4. Spectral measures with and without compact supports of arbitrary dimensions

4.1 Background 4.2 Main results

1. Spectrality of infinite convolutions (Background)

1. Spectrality of infinite convolutions (Background)

- A Borel probability measure μ on $\mathbb{R}^d$ is called a spectral measure if $\exists$ countable $\Lambda \subseteq \mathbb{R}^d$ s.t. $\{e^{2\pi i<\lambda,\cdot>} : \lambda \in \Lambda\}$ forms an orthonormal basis in $L^2(\mu)$.

1. Spectrality of infinite convolutions (Background)

- A Borel probability measure μ on $\mathbb{R}^d$ is called a spectral measure if $\exists$ countable $\Lambda \subseteq \mathbb{R}^d$ s.t. $\{e^{2\pi i<\lambda,\cdot>} : \lambda \in \Lambda\}$ forms an orthonormal basis in $L^2(\mu)$. We call Λ a spectrum of μ.

1. Spectrality of infinite convolutions (Background)

- A Borel probability measure μ on $\mathbb{R}^d$ is called a spectral measure if $\exists$ countable $\Lambda \subseteq \mathbb{R}^d$ s.t. $\{e^{2\pi i<\lambda,\cdot>} : \lambda \in \Lambda\}$ forms an orthonormal basis in $L^2(\mu)$. We call Λ a spectrum of μ.
- The existence of spectra of measures is a basic question in harmonic analysis since the orthonormal basis can be used for Fourier series expansions of functions in $L^2(\mu)$.

The spectrality of measures is widely studied (1974--Now).

The spectrality of measures is widely studied (1974--Now).

- Authors: An, Chen, Dai, Deng, Dong, Dutkay, Fu, Fuglede, Haussermann, He, Jorgensen, Łaba, Lai, Lau, Li, Liu, Lu, Luo, Miao, Pedersen, Sun, Wang, Wen, Wu, Yan, Zhang, ...

The spectrality of measures is widely studied (1974--Now).

- Authors: An, Chen, Dai, Deng, Dong, Dutkay, Fu, Fuglede, Haussermann, He, Jorgensen, Łaba, Lai, Lau, Li, Liu, Lu, Luo, Miao, Pedersen, Sun, Wang, Wen, Wu, Yan, Zhang, ...
- Journals: Adv. Math., Trans. Amer. Math. Soc., J. Math. Pures Appl., Appl. Comput. Harmon. Anal., J. Funct. Anal., Nonlinearity, ...

The spectrality of measures is widely studied (1974--Now).

- Authors: An, Chen, Dai, Deng, Dong, Dutkay, Fu, Fuglede, Haussermann, He, Jorgensen, Łaba, Lai, Lau, Li, Liu, Lu, Luo, Miao, Pedersen, Sun, Wang, Wen, Wu, Yan, Zhang, ...
- Journals: Adv. Math., Trans. Amer. Math. Soc., J. Math. Pures Appl., Appl. Comput. Harmon. Anal., J. Funct. Anal., Nonlinearity, ...

Before the work of Li-Miao-Wang (Adv. Math., 2022), spectral measures studied in the literature are compactly supported.

The spectrality of measures is widely studied (1974--Now).

- Authors: An, Chen, Dai, Deng, Dong, Dutkay, Fu, Fuglede, Haussermann, He, Jorgensen, Łaba, Lai, Lau, Li, Liu, Lu, Luo, Miao, Pedersen, Sun, Wang, Wen, Wu, Yan, Zhang, ...
- Journals: Adv. Math., Trans. Amer. Math. Soc., J. Math. Pures Appl., Appl. Comput. Harmon. Anal., J. Funct. Anal., Nonlinearity, ...

Before the work of Li-Miao-Wang (Adv. Math., 2022), spectral measures studied in the literature are compactly supported. They constructed singular spectral measures without compact supports in $\mathbb{R}$,

The spectrality of measures is widely studied (1974--Now).

- Authors: An, Chen, Dai, Deng, Dong, Dutkay, Fu, Fuglede, Haussermann, He, Jorgensen, Łaba, Lai, Lau, Li, Liu, Lu, Luo, Miao, Pedersen, Sun, Wang, Wen, Wu, Yan, Zhang, $\cdots$
- Journals: Adv. Math., Trans. Amer. Math. Soc., J. Math. Pures Appl., Appl. Comput. Harmon. Anal., J. Funct. Anal., Nonlinearity, $\cdots$

Before the work of Li-Miao-Wang (Adv. Math., 2022), spectral measures studied in the literature are compactly supported. They constructed singular spectral measures without compact supports in $\mathbb{R}$, and show that such measures are abundant in the sense that the dimension of their supports has the intermediate-value property.

Use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of Borel probability measures on $\mathbb{R}^d$.

Use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of Borel probability measures on $\mathbb{R}^d$.

Given $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$,

Use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of Borel probability measures on $\mathbb{R}^d$.

Given $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$, if the finite convolution

$$\mu_1 * \mu_2 * \cdots * \mu_n$$

converges weakly to a Borel probability measure,

Use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of Borel probability measures on $\mathbb{R}^d$.

Given $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$, if the finite convolution

$$\mu_1 * \mu_2 * \cdots * \mu_n$$

converges weakly to a Borel probability measure, we denote the weak limit measure by the infinite convolution

$$\mu_1 * \mu_2 * \mu_3 * \cdots$$

Use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of Borel probability measures on $\mathbb{R}^d$.

Given $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$, if the finite convolution

$$\mu_1 * \mu_2 * \cdots * \mu_n$$

converges weakly to a Borel probability measure, we denote the weak limit measure by the infinite convolution

$$\mu_1 * \mu_2 * \mu_3 * \cdots$$

and say that the infinite convolution exists.

Let δ_a denote the Dirac measure concentrated at the point a, Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \dfrac{1}{\# A} \sum_{a \in A} \delta_a$.

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A} \sum_{a \in A} \delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A} \sum_{a \in A} \delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$,

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A}\sum_{a\in A}\delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$, and B_k be a finite set of non-negative integers.

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A}\sum_{a\in A}\delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$, and B_k be a finite set of non-negative integers. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly consecutive digit sets w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{N_k\}_{k\geqslant 1}$,

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A}\sum_{a\in A}\delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$, and B_k be a finite set of non-negative integers. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly consecutive digit sets w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{N_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\} \pmod{N_k} \quad \text{for all } k \in \mathbb{N}$$

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A}\sum_{a\in A}\delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$, and B_k be a finite set of non-negative integers. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly consecutive digit sets w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{N_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\} \pmod{N_k} \quad \text{for all } k \in \mathbb{N}$$

and
$$\sum_{k=1}^{\infty}\frac{1}{m_k}\#(B_k \setminus \{0,1,\cdots,m_k-1\}) < \infty,$$

Let δ_a denote the Dirac measure concentrated at the point a, and for any non-empty finite set $A \subseteq \mathbb{R}^d$, define the uniform discrete measure supported on A by $\delta_A := \frac{1}{\# A} \sum_{a \in A} \delta_a$.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For each $k \in \mathbb{N}$, let m_k and N_k be integers satisfying $2 \leqslant m_k \leqslant N_k$ and $m_k \mid N_k$, and B_k be a finite set of non-negative integers. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly consecutive digit sets w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{N_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\} \pmod{N_k} \quad \text{for all } k \in \mathbb{N}$$

$$\text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{m_k} \#(B_k \setminus \{0,1,\cdots,m_k-1\}) < \infty,$$

then the infinite convolution

$$\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$$

exists and is a spectral measure with a spectrum in $\mathbb{Z}$.

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$.

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer,

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$,

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$, and $B_k \subseteq \mathbb{Z}^d$ be a finite set.

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$, and $B_k \subseteq \mathbb{Z}^d$ be a finite set. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{R_k\}_{k\geqslant 1}$,

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$, and $B_k \subseteq \mathbb{Z}^d$ be a finite set. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{R_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\}^d \pmod{R_k\mathbb{Z}^d} \quad \text{for all } k \in \mathbb{N}$$

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$, and $B_k \subseteq \mathbb{Z}^d$ be a finite set. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{R_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\}^d \pmod{R_k\mathbb{Z}^d} \quad \text{for all } k \in \mathbb{N}$$

and
$$\sum_{k=1}^{\infty} \frac{1}{m_k^d} \#(B_k \setminus \{0,1,\cdots,m_k-1\}^d) < \infty,$$

1. Spectrality of infinite convolutions (Main results)

Theorem 1.1 (L., submitted)

Let $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $m_k \geqslant 2$ be an integer, R_k be a $d \times d$ invertible integer matrix s.t. all entries of R_k are multiples of m_k and $[-m_k, m_k]^d \subseteq R_k^T[-1,1]^d$, and $B_k \subseteq \mathbb{Z}^d$ be a finite set. If $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w.r.t. $\{m_k\}_{k\geqslant 1}$ and $\{R_k\}_{k\geqslant 1}$, i.e.,

$$B_k \equiv \{0,1,\cdots,m_k-1\}^d \pmod{R_k\mathbb{Z}^d} \quad \text{for all } k \in \mathbb{N}$$

$$\text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{m_k^d} \#(B_k \setminus \{0,1,\cdots,m_k-1\}^d) < \infty,$$

then the infinite convolution

$$\mu = \delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots$$

exists and is a spectral measure with a spectrum in $\mathbb{Z}^d$.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions

1.1 Background 1.2 Main results

2. Supports of infinite convolutions

2.1 Background 2.2 Main results

3. Infinite sums of union sets

3.1 Background 3.2 Main results

4. Spectral measures with and without compact supports of arbitrary dimensions

4.1 Background 4.2 Main results

2. Supports of infinite convolutions (Background)

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\quad \operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?$

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum

$$\sum_{k=1}^{\infty} A_k := \left\{ \sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1 \right\}.$$

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\quad \operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum

$$\sum_{k=1}^{\infty} A_k := \Big\{ \sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1 \Big\}.$$

- It is well-known and widely used that: for any non-empty finite sets $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, if $\sum_{k=1}^{\infty} \max_{a \in A_k} |a| < \infty$,

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum

$$\sum_{k=1}^{\infty} A_k := \Big\{ \sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1 \Big\}.$$

- It is well-known and widely used that: for any non-empty finite sets $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, if $\sum_{k=1}^{\infty} \max_{a \in A_k} |a| < \infty$,

  then $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support,

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of
$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$
- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum
$$\sum_{k=1}^{\infty} A_k := \Big\{ \sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1 \Big\}.$$
- It is well-known and widely used that: for any non-empty finite sets $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, if $\sum_{k=1}^{\infty} \max_{a \in A_k} |a| < \infty$,
then $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support,
and $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \sum_{k=1}^{\infty} A_k$.

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?$ $\quad \operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ? \quad (1.1)$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum

$$\sum_{k=1}^{\infty} A_k := \Big\{ \sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1 \Big\}.$$

- It is well-known and widely used that: for any non-empty finite sets $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, if $\sum_{k=1}^{\infty} \max_{a \in A_k} |a| < \infty$,

  then $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support,

  and $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \sum_{k=1}^{\infty} A_k$.
- We will answer (1.1) in general,

2. Supports of infinite convolutions (Background)

- In the last section, we confirm the spectrality of a class of

$$\delta_{R_1^{-1}B_1} * \delta_{R_1^{-1}R_2^{-1}B_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}B_3} * \cdots.$$

- Question: $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = ?\quad \operatorname{spt}(\mu_1 * \mu_2 * \cdots) = ?\quad (1.1)$
- For $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, define the set of infinite sum

$$\sum_{k=1}^{\infty} A_k := \Big\{\sum_{k=1}^{\infty} a_k \text{ converges in } \mathbb{R}^d : a_k \in A_k, \forall k \geqslant 1\Big\}.$$

- It is well-known and widely used that: for any non-empty finite sets $A_1, A_2, \cdots \subseteq \mathbb{R}^d$, if $\sum_{k=1}^{\infty} \max_{a \in A_k} |a| < \infty$,

  then $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support, $\quad (1.2)$

  and $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \sum_{k=1}^{\infty} A_k$.
- We will answer (1.1) in general, and give a sufficient and neccessary condition for (1.2).

Notations and terminologies:

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$,

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j-th coordinate projection of A* from $\mathbb{R}^d$ to $\mathbb{R}$ by

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j*-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by

$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the j-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by

$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j*-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by

$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:

$$\inf A \in \mathbb{R} \cup \{-\infty\} \text{ and } \sup A \in \mathbb{R} \cup \{+\infty\} \text{ must exist;}$$

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j*-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by
$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:
$\inf A \in \mathbb{R} \cup \{-\infty\}$ and $\sup A \in \mathbb{R} \cup \{+\infty\}$ must exist;
write $\min A \in \mathbb{R}$ and $\max A \in \mathbb{R}$ if they exist.

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j*-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by

$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:

  $\inf A \in \mathbb{R} \cup \{-\infty\}$ and $\sup A \in \mathbb{R} \cup \{+\infty\}$ must exist;

  write $\min A \in \mathbb{R}$ and $\max A \in \mathbb{R}$ if they exist.

- Given $x_1, x_2, \cdots \in \mathbb{R} \cup \{\pm\infty\}$:

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the *j*-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by
$$(A)_j := \left\{ a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A \right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:
  $\inf A \in \mathbb{R} \cup \{-\infty\}$ and $\sup A \in \mathbb{R} \cup \{+\infty\}$ must exist;
  write $\min A \in \mathbb{R}$ and $\max A \in \mathbb{R}$ if they exist.

- Given $x_1, x_2, \cdots \in \mathbb{R} \cup \{\pm\infty\}$: we say that $\sum_{k=1}^{\infty} x_k$ converges if

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the j-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by
$$(A)_j := \left\{a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A\right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:
  $\inf A \in \mathbb{R} \cup \{-\infty\}$ and $\sup A \in \mathbb{R} \cup \{+\infty\}$ must exist;
  write $\min A \in \mathbb{R}$ and $\max A \in \mathbb{R}$ if they exist.

- Given $x_1, x_2, \cdots \in \mathbb{R} \cup \{\pm\infty\}$: we say that $\sum_{k=1}^{\infty} x_k$ converges if $x_k \in \mathbb{R}$ (not $\pm\infty$) for all $k \in \mathbb{N}$ and $\lim_{n \to \infty} \sum_{k=1}^{n} x_k$ exists (not $\pm\infty$);

Notations and terminologies:

- For all $A \subseteq \mathbb{R}^d$ and $j \in \{1, \cdots, d\}$, denote the j-th coordinate projection of A from $\mathbb{R}^d$ to $\mathbb{R}$ by
$$(A)_j := \left\{a_j \in \mathbb{R} : (a_1, \cdots, a_j, \cdots, a_d) \in A\right\}.$$

- For non-empty $A \subseteq \mathbb{R}$:

  $\inf A \in \mathbb{R} \cup \{-\infty\}$ and $\sup A \in \mathbb{R} \cup \{+\infty\}$ must exist;

  write $\min A \in \mathbb{R}$ and $\max A \in \mathbb{R}$ if they exist.

- Given $x_1, x_2, \cdots \in \mathbb{R} \cup \{\pm\infty\}$: we say that $\sum_{k=1}^{\infty} x_k$ converges if $x_k \in \mathbb{R}$ (not $\pm\infty$) for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists (not $\pm\infty$); otherwise, we say that $\sum_{k=1}^{\infty} x_k$ diverges.

2. Supports of infinite convolutions (Main results)

2. Supports of infinite convolutions (Main results)

Theorem 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$.

2. Supports of infinite convolutions (Main results)

Theorem 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$.

(1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$.

2. Supports of infinite convolutions (Main results)

Theorem 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$.

(1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$.

(2) $\mu_1 * \mu_2 * \cdots$ exists with compact support

$\Leftrightarrow$

2. Supports of infinite convolutions (Main results)

Theorem 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$.

(1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$.

(2) $\mu_1 * \mu_2 * \cdots$ exists with compact support

$\Leftrightarrow$ $\sum_{k=1}^{\infty} \inf(\operatorname{spt}\mu_k)_j$ and $\sum_{k=1}^{\infty} \sup(\operatorname{spt}\mu_k)_j$ converge $\forall j \in \{1, \cdots, d\}$

2. Supports of infinite convolutions (Main results)

Theorem 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots \in \mathcal{P}(\mathbb{R}^d)$.

(1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$.

(2) $\mu_1 * \mu_2 * \cdots$ exists with compact support

$\Leftrightarrow$ $\sum_{k=1}^{\infty} \inf(\operatorname{spt}\mu_k)_j$ and $\sum_{k=1}^{\infty} \sup(\operatorname{spt}\mu_k)_j$ converge $\forall j \in \{1, \cdots, d\}$

$\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \operatorname{spt}\mu_k$.

Corollary 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $A_1, A_2, \cdots \subseteq \mathbb{R}^d$ be non-empty finite sets.

Corollary 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $A_1, A_2, \cdots \subseteq \mathbb{R}^d$ be non-empty finite sets.

(1) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \overline{\sum_{k=1}^{\infty} A_k}$.

Corollary 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $A_1, A_2, \cdots \subseteq \mathbb{R}^d$ be non-empty finite sets.

(1) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \overline{\sum_{k=1}^{\infty} A_k}$.

(2) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support

$\Leftrightarrow$

Corollary 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $A_1, A_2, \cdots \subseteq \mathbb{R}^d$ be non-empty finite sets.

(1) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \overline{\sum_{k=1}^{\infty} A_k}$.

(2) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support

$\Leftrightarrow$ $\sum_{k=1}^{\infty} \min(A_k)_j$ and $\sum_{k=1}^{\infty} \max(A_k)_j$ converge $\forall j \in \{1, \cdots, d\}$,

Corollary 2.1 (L., submitted)

Let $d \in \mathbb{N}$ and $A_1, A_2, \cdots \subseteq \mathbb{R}^d$ be non-empty finite sets.

(1) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \overline{\sum_{k=1}^{\infty} A_k}$.

(2) $\delta_{A_1} * \delta_{A_2} * \cdots$ exists with compact support

$\Leftrightarrow$ $\sum_{k=1}^{\infty} \min(A_k)_j$ and $\sum_{k=1}^{\infty} \max(A_k)_j$ converge $\forall j \in \{1, \cdots, d\}$,

$\Rightarrow$ $\mathrm{spt}(\delta_{A_1} * \delta_{A_2} * \cdots) = \sum_{k=1}^{\infty} A_k$.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background   1.2 Main results

2. Supports of infinite convolutions
   2.1 Background   2.2 Main results

3. **Infinite sums of union sets**
   3.1 Background   3.2 Main results

4. Spectral measures with and without compact supports of arbitrary dimensions
   4.1 Background   4.2 Main results

3. Infinite sums of union sets (Background)

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt} \mu_k}$;

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

(1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \mathrm{spt}\mu_k}$;

(2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \mathrm{spt}\mu_k$.

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \mathrm{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \mathrm{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ is not closed.

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \mathrm{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \mathrm{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ is not closed.

- Question: When is $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ closed?

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \operatorname{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not closed.
- Question: When is $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ closed?

(especially when $\operatorname{spt}(\mu_1 * \mu_2 * \cdots)$ is not compact,

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \mathrm{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\mathrm{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \mathrm{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ is not closed.
- Question: When is $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ closed?

(especially when $\mathrm{spt}(\mu_1 * \mu_2 * \cdots)$ is not compact, i.e., $\sum_{k=1}^{\infty} \mathrm{spt}\mu_k$ is not bounded.)

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \operatorname{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not closed.

- Question: When is $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ closed?

  $\Big($especially when $\operatorname{spt}(\mu_1 * \mu_2 * \cdots)$ is not compact,

  i.e., $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not bounded.$\Big)$

- Therefore, in the following we study $\sum_{k=1}^{\infty} (A_k \cup A'_k)$,

  where $\{A'_k\}_{k\geqslant 1}$ are far from the original in some sense.

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \operatorname{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not closed.

- Question: When is $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ closed?

  (especially when $\operatorname{spt}(\mu_1 * \mu_2 * \cdots)$ is not compact, i.e., $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not bounded.)

- Therefore, in the following we study $\sum_{k=1}^{\infty} (A_k \cup A_k')$, where $\{A_k'\}_{k \geqslant 1}$ are far from the original in some sense.

When is $\sum_{k=1}^{\infty} (A_k \cup A_k')$ closed?

3. Infinite sums of union sets (Background)

- By Theorem 2.1:

  (1) $\mu_1 * \mu_2 * \cdots$ exists $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \overline{\sum_{k=1}^{\infty} \operatorname{spt}\mu_k}$;

  (2) $\mu_1 * \mu_2 * \cdots$ exists with compact support $\Rightarrow$ $\operatorname{spt}(\mu_1 * \mu_2 * \cdots) = \sum_{k=1}^{\infty} \operatorname{spt}\mu_k$.

- $\exists$ example s.t. $\mu_1 * \mu_2 * \cdots$ exists but $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not closed.

- Question: When is $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ closed?

  (especially when $\operatorname{spt}(\mu_1 * \mu_2 * \cdots)$ is not compact, i.e., $\sum_{k=1}^{\infty} \operatorname{spt}\mu_k$ is not bounded.)

- Therefore, in the following we study $\sum_{k=1}^{\infty} (A_k \cup A'_k)$, where $\{A'_k\}_{k\geqslant 1}$ are far from the original in some sense.

When is $\sum_{k=1}^{\infty} (A_k \cup A'_k)$ closed? Fractal dimensions of $\sum_{k=1}^{\infty} (A_k \cup A'_k)$?

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

(1)

then $\sum_{k=1}^{\infty}(A_k \cup A_k')$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A_k' \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1)

then $\sum_{k=1}^{\infty}(A_k \cup A_k')$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A'_k$ is closed $\forall k \in \mathbb{N}$,

then $\sum_{k=1}^{\infty}(A_k \cup A'_k)$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A'_k$ is closed $\forall k \in \mathbb{N}$, and $\forall j \in \{1, \cdots, d\}$

both $\sum_{k=1}^{\infty} \inf(A_k)_j$ and $\sum_{k=1}^{\infty} \sup(A_k)_j$ converge

then $\sum_{k=1}^{\infty} (A_k \cup A'_k)$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A_k' \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A_k'$ is closed $\forall k \in \mathbb{N}$, and $\forall j \in \{1, \cdots, d\}$

both $\sum_{k=1}^{\infty} \inf(A_k)_j$ and $\sum_{k=1}^{\infty} \sup(A_k)_j$ converge

and $\inf_{k \in \mathbb{N}} \inf(A_k')_j > -\infty$ or $\sup_{k \in \mathbb{N}} \sup(A_k')_j < +\infty$.

then $\sum_{k=1}^{\infty}(A_k \cup A_k')$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A'_k$ is closed $\forall k \in \mathbb{N}$, and $\forall j \in \{1, \cdots, d\}$

both $\sum_{k=1}^{\infty} \inf(A_k)_j$ and $\sum_{k=1}^{\infty} \sup(A_k)_j$ converge

and $\inf_{k \in \mathbb{N}} \inf(A'_k)_j > -\infty$ or $\sup_{k \in \mathbb{N}} \sup(A'_k)_j < +\infty$.

If $\liminf_{k \to \infty} \inf_{a \in A'_k} |a| = +\infty$, then $\sum_{k=1}^{\infty} (A_k \cup A'_k)$ is closed.

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A'_k$ is closed $\forall k \in \mathbb{N}$, and $\forall j \in \{1, \cdots, d\}$

$$\text{both } \sum_{k=1}^{\infty} \inf(A_k)_j \text{ and } \sum_{k=1}^{\infty} \sup(A_k)_j \text{ converge}$$

and $$\inf_{k \in \mathbb{N}} \inf(A'_k)_j > -\infty \quad \text{or} \quad \sup_{k \in \mathbb{N}} \sup(A'_k)_j < +\infty. \qquad (\star)$$

If $\liminf_{k \to \infty} \inf_{a \in A'_k} |a| = +\infty$, then $\sum_{k=1}^{\infty} (A_k \cup A'_k)$ is closed.

$$(\star) \iff \exists\, c_1, c_2 \in \mathbb{R} \text{ and } X_1, \cdots, X_d \in \left\{ [c_1, +\infty), (-\infty, c_2] \right\}$$

$$\text{s.t. } \bigcup_{k=1}^{\infty} A'_k \subseteq X_1 \times \cdots \times X_d$$

3. Infinite sums of union sets (Main results)

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(1) Suppose that $A_k \cup A'_k$ is closed $\forall k \in \mathbb{N}$, and $\forall j \in \{1, \cdots, d\}$

both $\sum_{k=1}^{\infty} \inf(A_k)_j$ and $\sum_{k=1}^{\infty} \sup(A_k)_j$ converge

and $\inf_{k\in\mathbb{N}} \inf(A'_k)_j > -\infty$ or $\sup_{k\in\mathbb{N}} \sup(A'_k)_j < +\infty$. $(\star)$

If $\liminf_{k\to\infty} \inf_{a\in A'_k} |a| = +\infty$, then $\sum_{k=1}^{\infty}(A_k \cup A'_k)$ is closed.

$(\star) \iff \exists\, c_1, c_2 \in \mathbb{R}$ and $X_1, \cdots, X_d \in \left\{[c_1, +\infty), (-\infty, c_2]\right\}$

s.t. $\bigcup_{k=1}^{\infty} A'_k \subseteq X_1 \times \cdots \times X_d$

$\Leftarrow \bigcup_{k=1}^{\infty} A'_k \subseteq [0, +\infty)^d$ or $(-\infty, 0]^d$.

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A_k' \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2)

$$\dim_H \sum_{k=1}^{\infty}\left(A_k \cup A_k'\right) = ?$$

$$\dim_P \sum_{k=1}^{\infty}\left(A_k \cup A_k'\right) = ?$$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A'_k is at most countable for $\forall k \in \mathbb{N}$.

$$\dim_H \sum_{k=1}^{\infty} \left(A_k \cup A'_k\right) = ?$$

$$\dim_P \sum_{k=1}^{\infty} \left(A_k \cup A'_k\right) = ?$$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A'_k is at most countable for $\forall k \in \mathbb{N}$.

If $\varliminf_{k\to\infty} \inf_{a \in A'_k} |a| > 0$,

$$\dim_H \sum_{k=1}^{\infty} \left(A_k \cup A'_k \right) = ?$$

$$\dim_P \sum_{k=1}^{\infty} \left(A_k \cup A'_k \right) = ?$$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A'_k is at most countable for $\forall k \in \mathbb{N}$.

If $\varliminf_{k\to\infty} \inf_{a \in A'_k} |a| > 0$, then

① $\mathcal{H}^s\left(\sum_{k=1}^{\infty}(A_k \cup A'_k)\right) = 0$ iff $\mathcal{H}^s\left(\sum_{k=1}^{\infty} A_k\right) = 0$ for $\forall s \in [0, d]$,

$$\dim_H \sum_{k=1}^{\infty}\left(A_k \cup A'_k\right) = ?$$

$$\dim_P \sum_{k=1}^{\infty}\left(A_k \cup A'_k\right) = ?$$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A_k' \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A_k' is at most countable for $\forall k \in \mathbb{N}$.

If $\varliminf_{k\to\infty} \inf_{a \in A_k'} |a| > 0$, then

① $\mathcal{H}^s\left(\sum_{k=1}^{\infty}(A_k \cup A_k')\right) = 0$ iff $\mathcal{H}^s\left(\sum_{k=1}^{\infty} A_k\right) = 0$ for $\forall s \in [0, d]$,

and $\dim_H \sum_{k=1}^{\infty}\left(A_k \cup A_k'\right) = \dim_H \sum_{k=1}^{\infty} A_k$;

$$\dim_P \sum_{k=1}^{\infty}\left(A_k \cup A_k'\right) = ?$$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A'_k \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A'_k is at most countable for $\forall k \in \mathbb{N}$.

If $\varliminf_{k\to\infty} \inf_{a \in A'_k} |a| > 0$, then

① $\mathcal{H}^s\left(\sum_{k=1}^{\infty}(A_k \cup A'_k)\right) = 0$ iff $\mathcal{H}^s\left(\sum_{k=1}^{\infty} A_k\right) = 0$ for $\forall s \in [0, d]$,

and $\dim_H \sum_{k=1}^{\infty}\left(A_k \cup A'_k\right) = \dim_H \sum_{k=1}^{\infty} A_k$;

② $\mathcal{P}^s\left(\sum_{k=1}^{\infty}(A_k \cup A'_k)\right) = 0$ iff $\mathcal{P}^s\left(\sum_{k=1}^{\infty} A_k\right) = 0$ for $\forall s \in [0, d]$,

$\dim_P \sum_{k=1}^{\infty}\left(A_k \cup A'_k\right) = ?$

Theorem 3.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $A_k, A_k' \subseteq \mathbb{R}^d$ where $A_k \neq \varnothing$.

(2) Suppose that A_k' is at most countable for $\forall k \in \mathbb{N}$.

If $\varliminf_{k\to\infty} \inf_{a \in A_k'} |a| > 0$, then

① $\mathcal{H}^s\Big(\sum_{k=1}^{\infty}(A_k \cup A_k')\Big) = 0$ iff $\mathcal{H}^s\Big(\sum_{k=1}^{\infty} A_k\Big) = 0$ for $\forall s \in [0, d]$,

and $\dim_H \sum_{k=1}^{\infty}\big(A_k \cup A_k'\big) = \dim_H \sum_{k=1}^{\infty} A_k$;

② $\mathcal{P}^s\Big(\sum_{k=1}^{\infty}(A_k \cup A_k')\Big) = 0$ iff $\mathcal{P}^s\Big(\sum_{k=1}^{\infty} A_k\Big) = 0$ for $\forall s \in [0, d]$,

and $\dim_P \sum_{k=1}^{\infty}\big(A_k \cup A_k'\big) = \dim_P \sum_{k=1}^{\infty} A_k$.

Spectrality and supports of infinite convolutions in $\mathbb{R}^d$

1. Spectrality of infinite convolutions
   1.1 Background 1.2 Main results
2. Supports of infinite convolutions
   2.1 Background 2.2 Main results
3. Infinite sums of union sets
   3.1 Background 3.2 Main results
4. **Spectral measures with and without compact supports of arbitrary dimensions**
   4.1 Background 4.2 Main results

4. Spectral measures with and without compact supports of arbitrary dimensions (Background)

4. Spectral measures with and without compact supports of arbitrary dimensions (Background)

The Hausdorff dimension and packing dimension of the supports of singular spectral measures without compact supports in $\mathbb{R}$ have the intermediate-value property.

4. Spectral measures with and without compact supports of arbitrary dimensions (Background)

The Hausdorff dimension and packing dimension of the supports of singular spectral measures without compact supports in $\mathbb{R}$ have the intermediate-value property.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For any $0 \leqslant \alpha \leqslant \beta \leqslant 1$,

4. Spectral measures with and without compact supports of arbitrary dimensions (Background)

The Hausdorff dimension and packing dimension of the supports of singular spectral measures without compact supports in $\mathbb{R}$ have the intermediate-value property.

Theorem (Li-Miao-Wang, Adv. Math., 2022)

For any $0 \leqslant \alpha \leqslant \beta \leqslant 1$, there exists a singular spectral measure μ on $\mathbb{R}$ without compact support s.t.

$$\dim_H \operatorname{spt}\mu = \alpha \leqslant \beta = \dim_P \operatorname{spt}\mu.$$

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose $\displaystyle\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose $$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$$

$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose
$$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1\cdots N_k} = +\infty,$$
$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$
and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$.

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose
$$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$$
$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$
and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$. Then $\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$ exists,

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \emptyset$.

Suppose $$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$$
$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$
and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$. Then $\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$ exists, is a singular spectral measure,

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose
$$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1\cdots N_k} = +\infty,$$
$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$
and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$. Then $\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$ exists, is a singular spectral measure, $\mathrm{spt}\mu = \Sigma_{k=1}^{\infty} N_1^{-1}\cdots N_k^{-1}B_k$,

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \emptyset$.

Suppose $$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$$

$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$

and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$. Then $\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$ exists, is a singular spectral measure, $\mathrm{spt}\mu = \Sigma_{k=1}^{\infty} N_1^{-1} \cdots N_k^{-1} B_k$,

$$\dim_H \mathrm{spt}\mu = \varliminf_{k\to\infty} \frac{\log \# G_1 \cdots \# G_k}{\log N_1 \cdots N_k}$$

4. Spectral measures with and without compact supports of arbitrary dimensions (Main results)

First, by Theorem 1.1, Corollary 2.2 (1) and Theorem 3.1 we can prove:

Corollary 4.1 (L., submitted)

Let $d \in \mathbb{N}$. For $\forall k \in \mathbb{N}$, let $N_k \geqslant m_k \geqslant 2$ be integers with $m_k \mid N_k$ and $B_k \subseteq \{0,1,2,\cdots\}^d$ be a finite set with $G_k := B_k \cap \{0,1,\cdots,m_k-1\}^d \neq \varnothing$.

Suppose $$\lim_{k\to\infty} \frac{\min\{|x| : x \in B_k \setminus \{0,1,\cdots,m_k-1\}^d\}}{N_1 \cdots N_k} = +\infty,$$
$$\lim_{k\to\infty} \frac{\log N_k}{\log N_1 \cdots N_k} = 0 \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\# G_k}{N_k^d} = 0,$$
and suppose that $\{B_k\}_{k\geqslant 1}$ is a sequence of nearly d-th power lattices w. r. t. $\{m_k\}_{k\geqslant 1}$ and $\{\mathrm{diag}(N_k,\cdots,N_k)\}_{k\geqslant 1}$. Then $\mu = \delta_{N_1^{-1}B_1} * \delta_{N_1^{-1}N_2^{-1}B_2} * \delta_{N_1^{-1}N_2^{-1}N_3^{-1}B_3} * \cdots$ exists, is a singular spectral measure, $\mathrm{spt}\mu = \Sigma_{k=1}^{\infty} N_1^{-1} \cdots N_k^{-1} B_k$,
$$\dim_H \mathrm{spt}\mu = \varliminf_{k\to\infty} \frac{\log \# G_1 \cdots \# G_k}{\log N_1 \cdots N_k} \quad \text{and} \quad \dim_P \mathrm{spt}\mu = \varlimsup_{k\to\infty} \frac{\log \# G_1 \cdots \# G_k}{\log N_1 \cdots N_k}.$$

Corollary 4.2 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$,

Corollary 4.3 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$,

Corollary 4.2 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$, there exists a singular spectral measure μ on $\mathbb{R}^d$ with compact support s.t.

$$\dim_H \operatorname{spt}\mu = \alpha \leqslant \beta = \dim_P \operatorname{spt}\mu.$$

Corollary 4.3 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$,

Corollary 4.2 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$, there exists a singular spectral measure μ on $\mathbb{R}^d$ with compact support s.t.

$$\dim_H \operatorname{spt}\mu = \alpha \leqslant \beta = \dim_P \operatorname{spt}\mu.$$

Corollary 4.3 (L., submitted)

Let $d \in \mathbb{N}$. For any $0 \leqslant \alpha \leqslant \beta \leqslant d$, there exists a singular spectral measure μ on $\mathbb{R}^d$ without compact support s.t.

$$\dim_H \operatorname{spt}\mu = \alpha \leqslant \beta = \dim_P \operatorname{spt}\mu.$$

Thank you!