# Complex continued fractions and Kleinian circle packings

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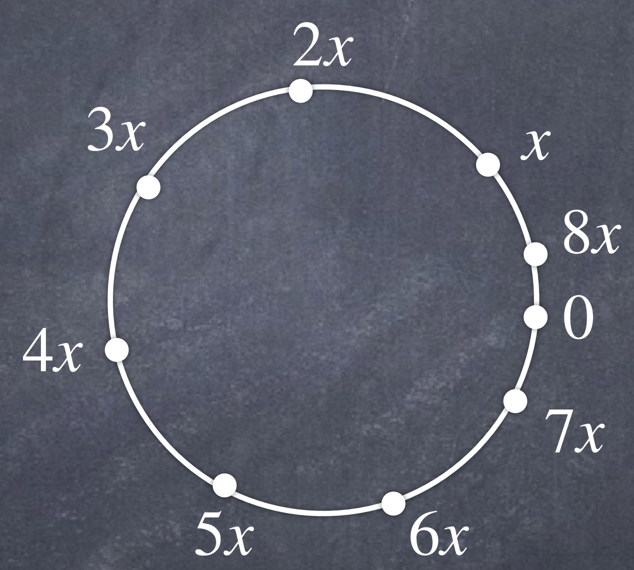
# I. Continued fractions (classical)

Continued fraction: $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$, $x \in [0,1)$. $\quad \dfrac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}$

best approximating rationals.

Dirichlet Thm (Pigeonhole principle) : $\forall N \in \mathbb{N}, \exists p, q \in \mathbb{Z}$ s.t. $0 < q < N$,

$$|qx - p| < \frac{1}{N} < \frac{1}{q}.$$




Gauss map: $T(x) := \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor$ : shift map on $\{(a_i)\}$.

# Relation to homogeneous dynamics of $H \leq SL_2(\mathbb{R})$ acting on $X = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$

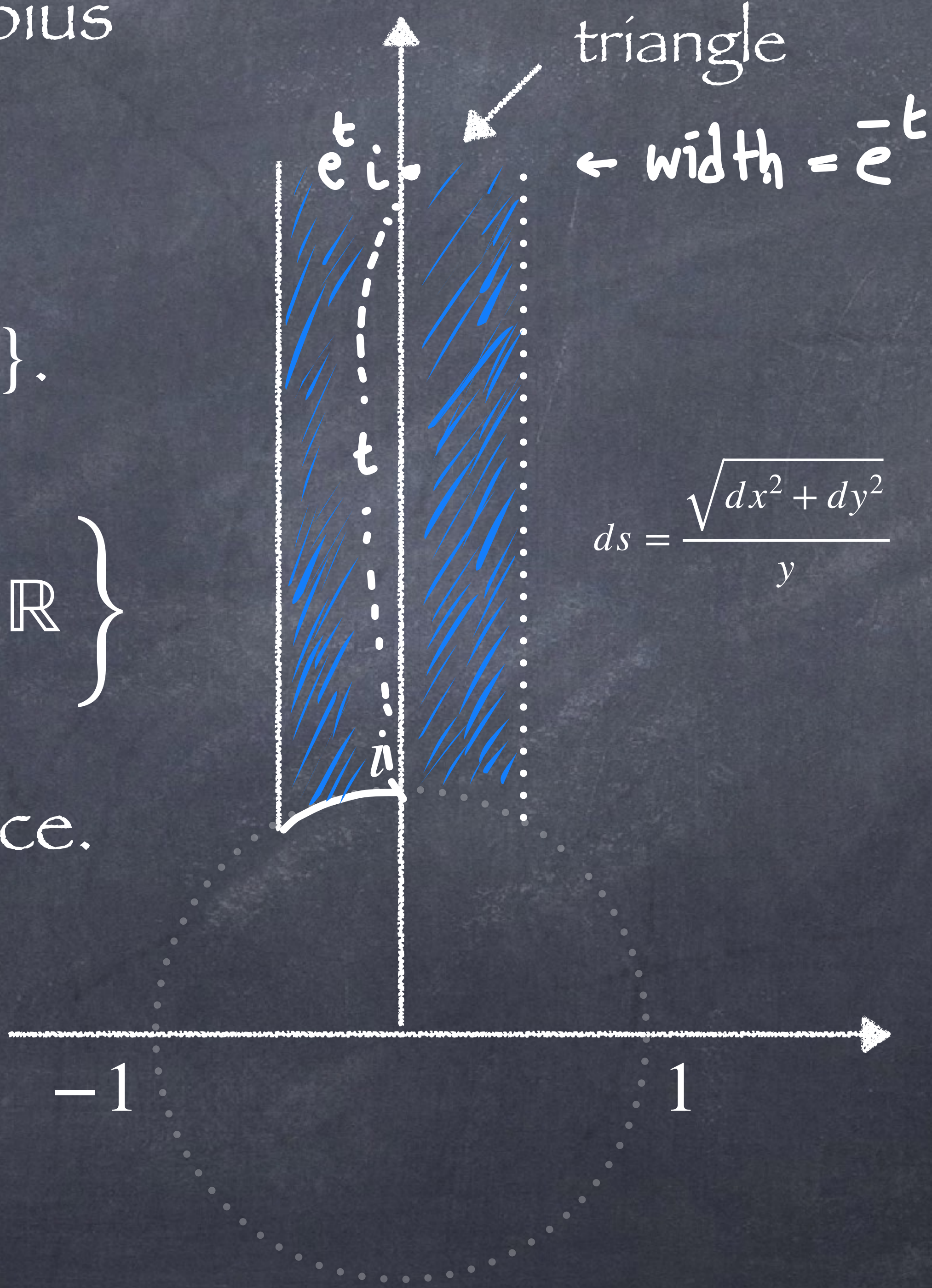
(1) $X \simeq T^1\mathbb{H}^2/SL_2(\mathbb{Z})$: $SL_2(\mathbb{R})$ acts by Mobius transform (isometries):

$$z \mapsto \frac{az+b}{cz+d}, \quad z \in \mathbb{H}^2 := \{x+iy : y > 0\}.$$

The action of $H = \left\{ A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$

is the geodesic flow on the modular surface.

(curvature $= -1$)



# Continued fraction and geodesic flow

1. $a_i$ ~# triangles  crossed by the $i^{th}$ excursion of geodesic with endpt x, separated by hitting "compact part".

2. Height of the $i^{th}$ excursion $\sim \log \dfrac{a_i}{2}$ .

3. Diophantine property of $x$ <-> property of $a_i$'s.

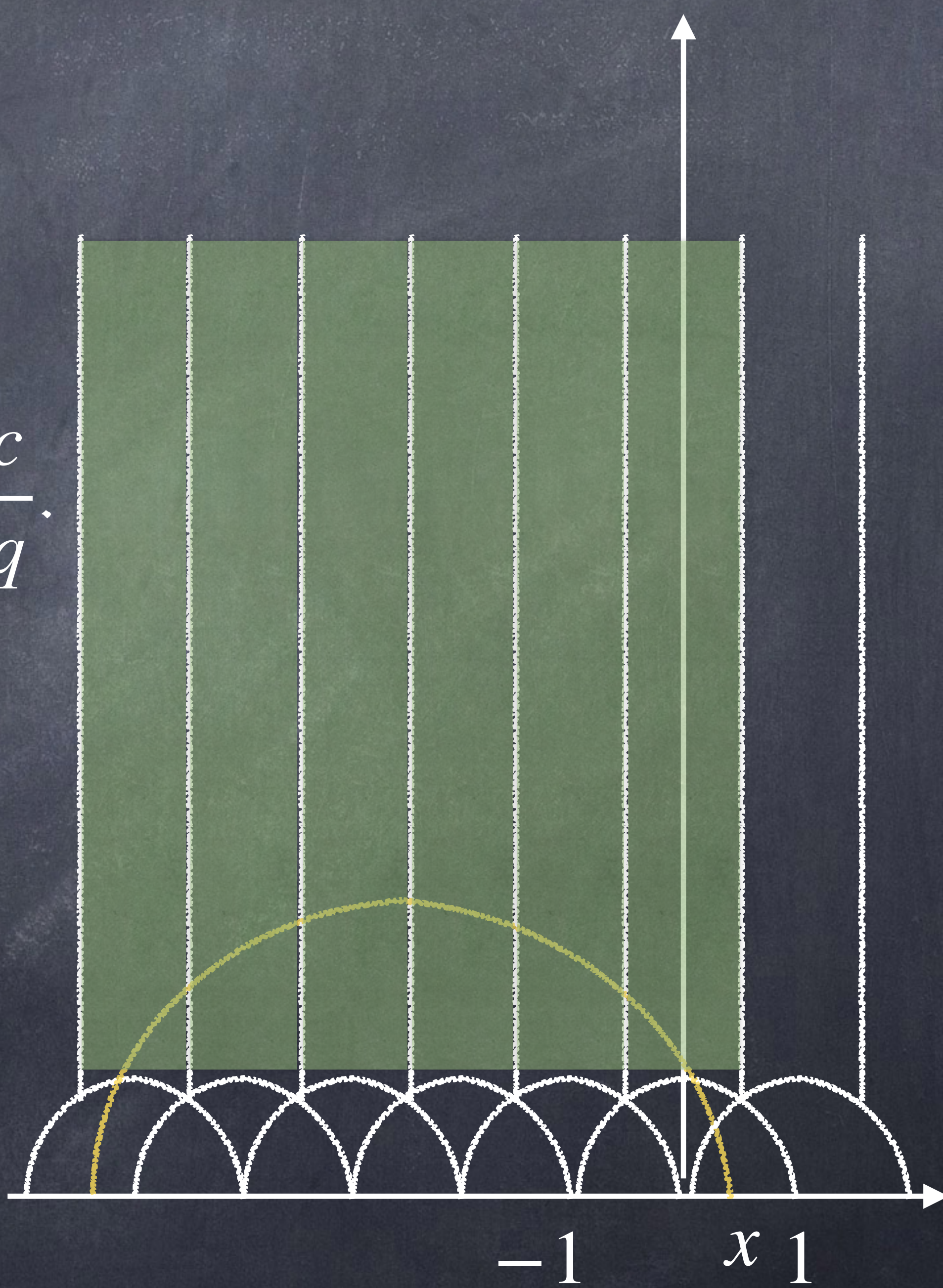
E.g. $x$ is <u>badly approximable</u> if $\exists c > 0$ s.t. $|qx - p| > \dfrac{c}{q}$.

iff $a_n$ is bounded.

iff geodesic stays in a cpt part.

Geodesic flow is ergodic => $\mu_{Leb}$(Bad) =0.

= { badly approxim }



Generalization of real CF and geodesic flow:

(1) Neg. curved metric spaces with "Anosov flows"

$\mathbb{H}^n$, Trees, Riemannian manifolds $(\kappa < 0)$,

various $\Gamma$'s (various CF's : w/ Jaelin Kim, Seulbee Lee )

In homogeneous dynamics:

(2) $X = T^1\mathbb{H}^2/SL_2(\mathbb{Z}) \simeq SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \simeq$ {lattices of area 1}

(3) $G/\Gamma, K\backslash G/\Gamma$ for discrete gp $\Gamma$ in a ss alg. gp $G$ $(\kappa \leq 0)$

Bruhat-Tits buildings (w/ Jaeyong Kim, Sanghoon Kwon)

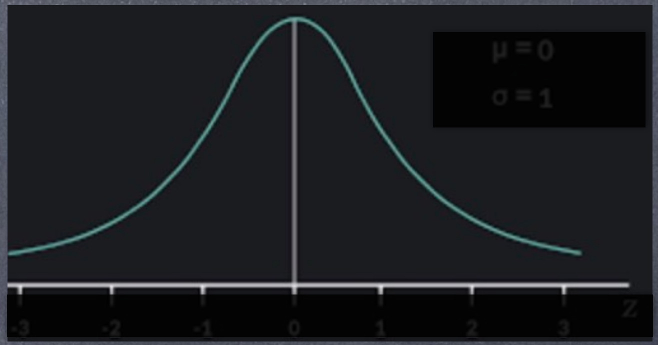
Complex continued fraction (Jungwon Lee-Dohyeong Kim-L.)

Kleinian circle packings (Kangrae Park-L.-Yongquan Zhang)

Q. Distribution of length of continued fraction on $\{\frac{p}{q} : p \leq q, gcd(p, q = 1\}$?

On $\Omega_N := \left\{\frac{p}{q} : gcd(p,q) = 1, q \leq N\right\}$?

Thm (Baladi-Vallée) Asymptotically Gaussian as $N \to \infty$ .



Gaussian

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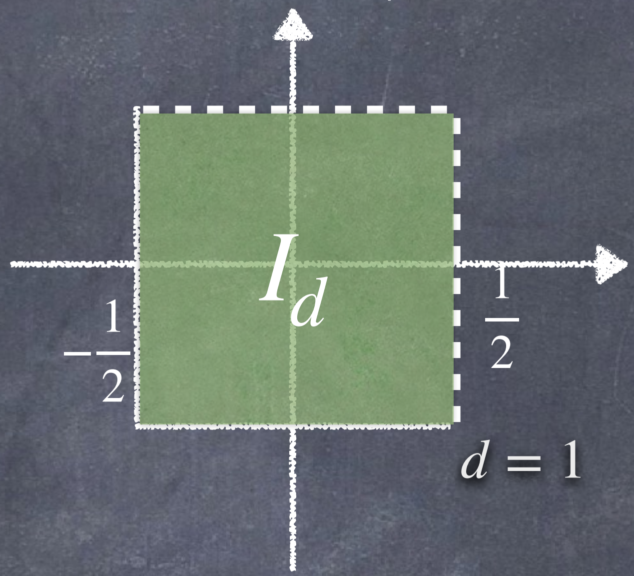
Thm (Baladi-Vallée) Asymptotically Gaussian as $N \to \infty$.

# II. Hurwitz continued fraction

Hurwitz CF  Replace $\mathbb{Z}$ by $\mathbb{Z}[\sqrt{-d}]$, e.g. $\mathbb{Z}[i]$.

Euclidean iff d=1, 2, 3, 7, 11.  Replace [0,1) by $I_d$ "fund domain" of $\mathbb{Z}[\sqrt{-d}]$

$[z]_d$ : nearest integer iff $z - [z]_d \in I_d$. $z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$




Gauss map: $T(z) = \frac{1}{z} - \left[\frac{1}{z}\right]_d$ : shift map on $(a_i)$

Thm 1 (Dohyeong Kim-Jungwon Lee- L.) Length of the Hurwitz CF on $\Omega_N := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}[\sqrt{-d}], gcd(p,q) = 1, |q|^2 \leq N \right\}$ is asymptotically Gaussian.

# Hurwitz nearest integer CF (cf. Dani: arbitrary $\mathbb{C}$-CF)

- Gauss map ergodicity : Ei-Nakada-Natsui
- Dioph approx. : Hensley, Bugeaud-Gonzalez-Robert-Hussain

New phenomenon: each branch is not surjective.

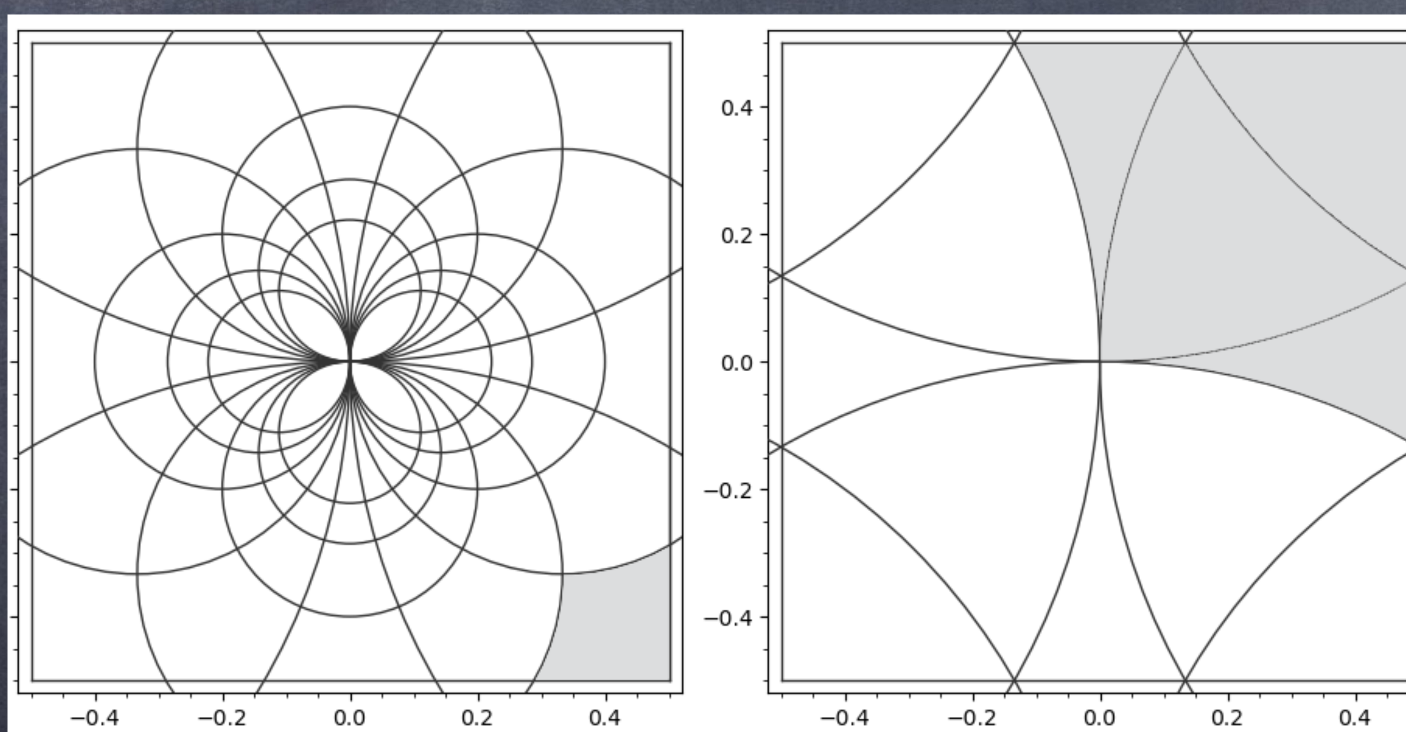
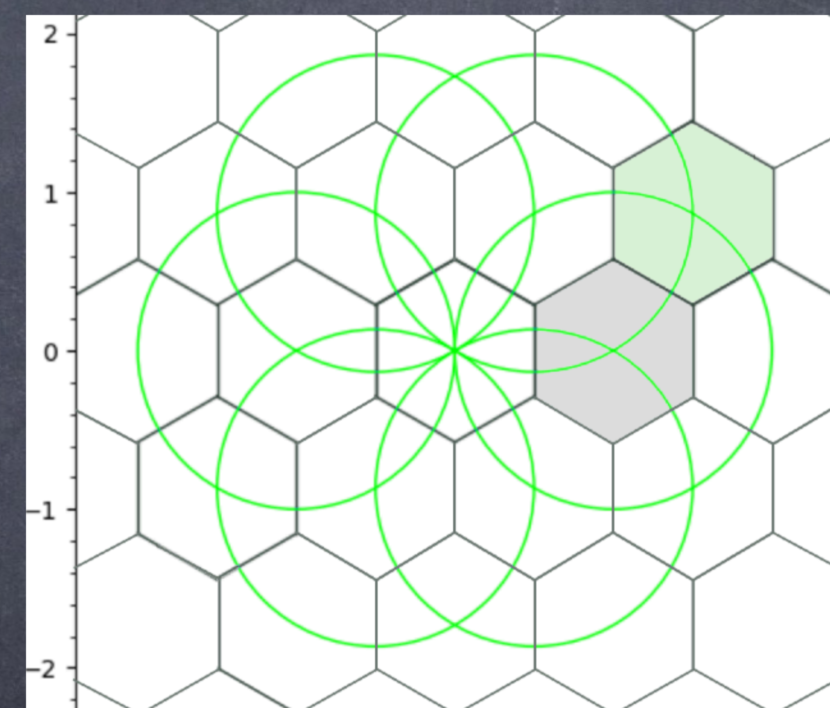


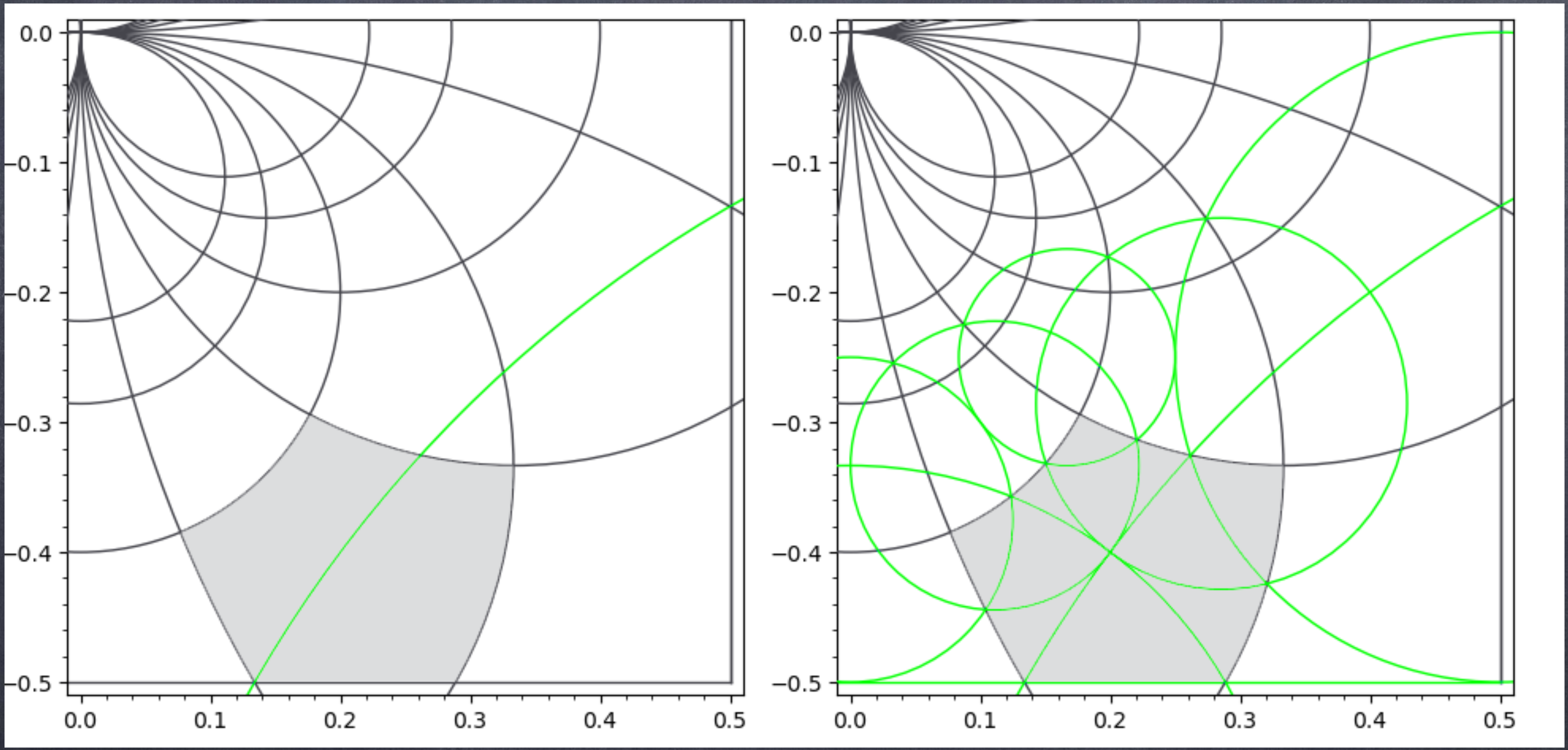

Image of some "cylinder" (d=1)




d=3

Thm (Ei-Nakada-Natsui) proved Finite range property, which implies that there is a finite partition of $I_d$ stable under T:

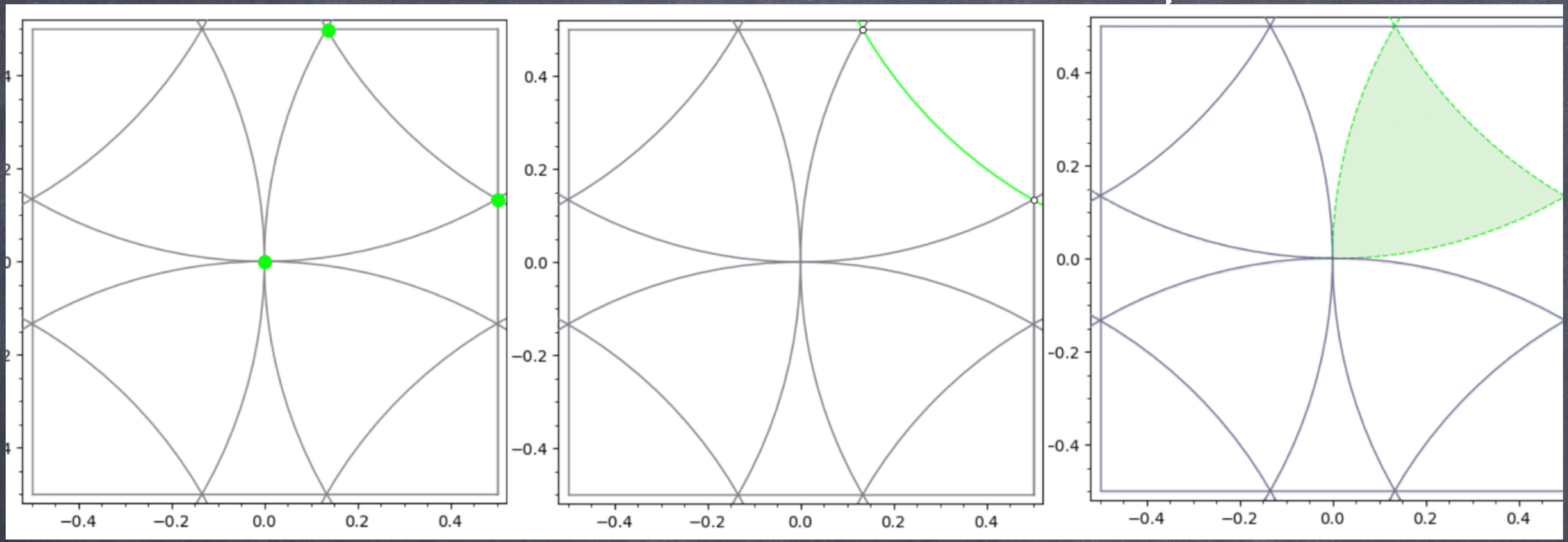
the image of each branch is the union of cells.




Stability: each “cylinder” is subdivided with similar partitions (d=1)

Ei-Nakada-Natsui proved Finite range property, which implies that there is a finite partition of $I_d$ stable under T.

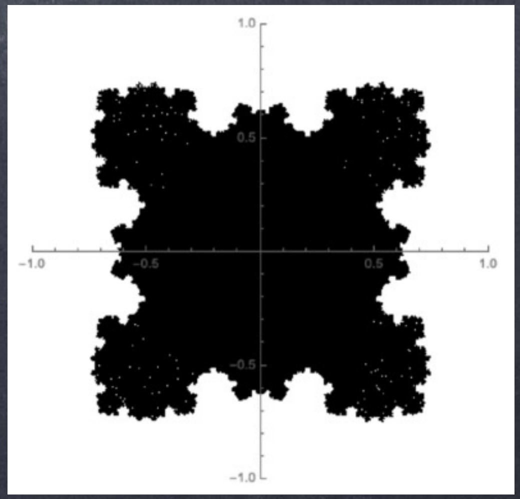
Modification for us: add vertices and 1-cells (simplicial cx)




Cells of the partition of I_d (d=1)

New function space $\bigoplus_{P \in \mathscr{P}} C^1(P)$

Another challenge: dual system is mysterious.




Ei-Nakada-Natsui : natural extension (&dual system)

Thm2 (KLL) (mod p equidistribution) Values of $\ell \pmod{p}$ equidistributes: $\forall a \in \mathbb{Z}/q\mathbb{Z}$,

$$\mathbb{P}_N[\ell \equiv a \pmod{p} : \Omega_N] = \frac{1}{p} + o(1)\,.$$

Motivation:

J.Lee-Haesang Sun mod p non-vanishing of average of special twisted L-values using mod p equidistribution of modular symbols.

In progress: Use mod p equidistribution of Bianchi modular symbols to prove analogous result. Need to define period.

(Cf. Real case: Vatsal's canonical period)

# Dynamics behind continued fractions

Thm 2 (KLL) Central Limit Thm (continuous version)

$$\mathbb{P}\left[\frac{\ell_n - \widehat{\mu}(c)n}{\widehat{\delta}(c)\sqrt{n}} \leq u\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{t^2}{2}}dt + O\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$, where $\mathbb{P}$ :proba on $I_d$ with $C^1$-invariant density.

The expectation and variance satisfy

$$\mathbb{E}[C_n] = \widehat{\mu}(c)n + \widehat{\mu}_1(c) + O(\theta^n),\ \ \mathbb{V}[C_n] = \widehat{\delta}(c)n + \widehat{\delta}_1(c) + O(\theta^n)$$

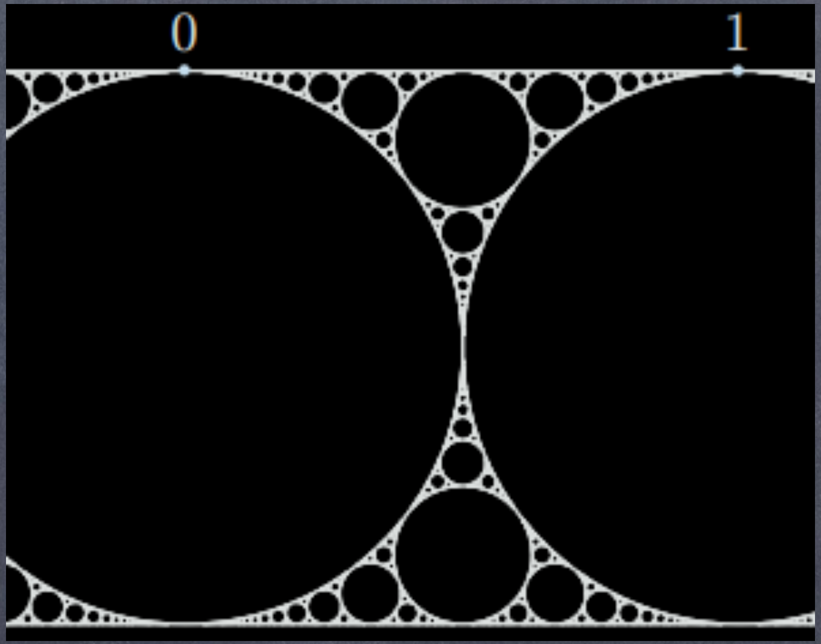
for some $\theta < 1$, $\widehat{\mu}_1(c) > 0$, $\widehat{\delta}_1(c) > 0$. (c: cost function =1 for length.)

Proof. Spectral gap of <u>Transfer operator $\mathscr{L}$</u> associated to Gauss map

2-dim'l van der Corput Lemma (uses dual system)

# III. Kleinian circle packings

Another variant of Complex CF by Asmus Schmidt gives rise to the Apollonian group. Its limit set is a circle packing:




Q. Diophantine approximation on Apollonian circle packings?

(Approximating limit points by tangency points of circles)

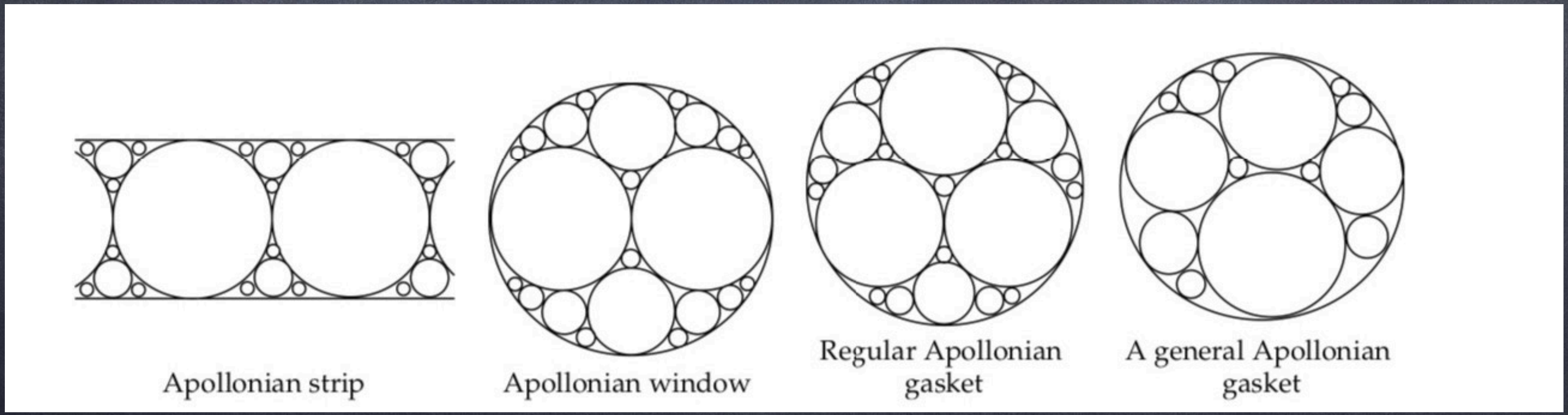
# III. Kleinian circle packings

Circle packing = Union of circles in a region s.t.

any two circles intersect only tangentially.

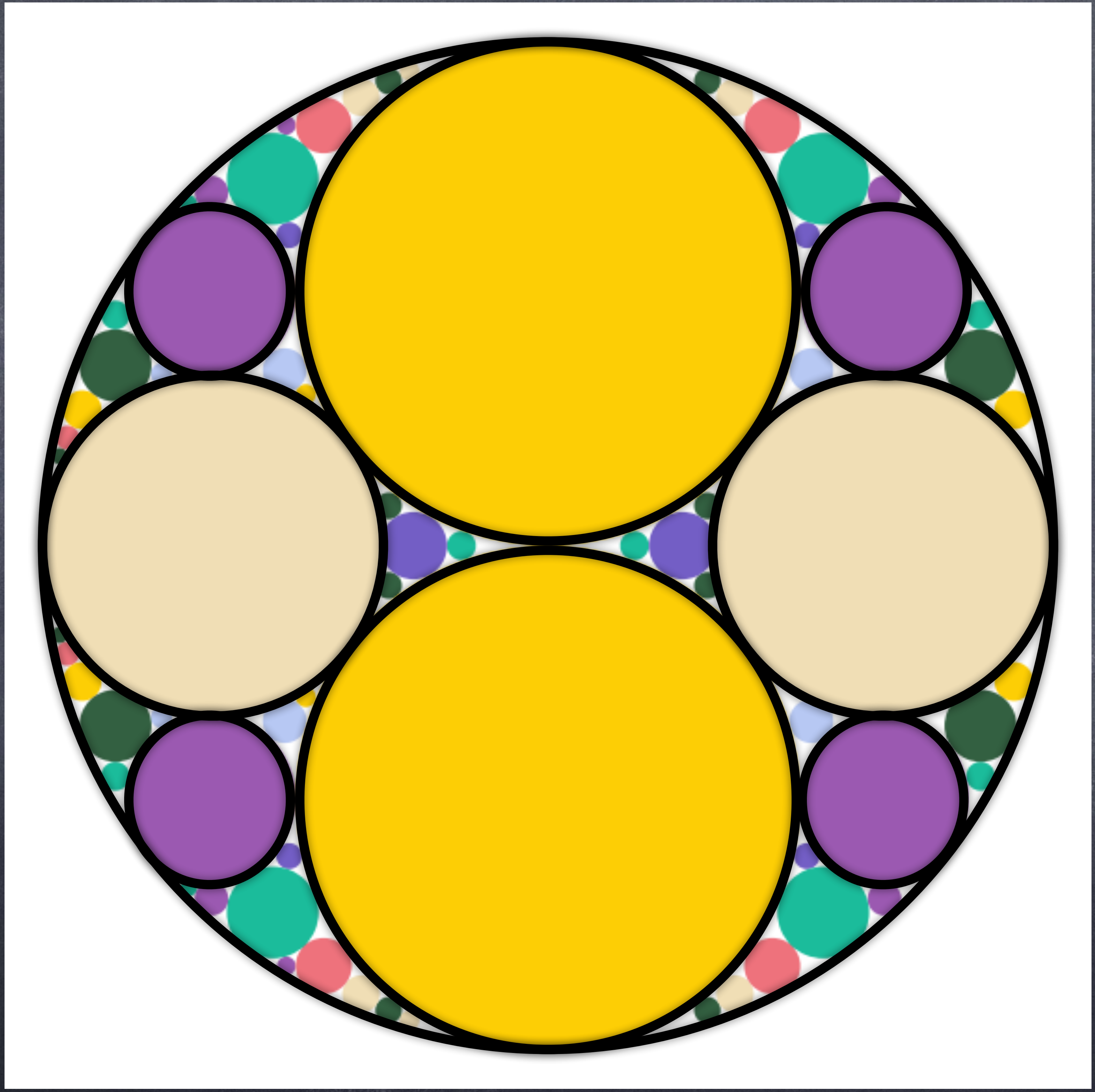
Kleinian Circle Packing : = limit set of Kleinian gp

e.g. Apollonian circle packings

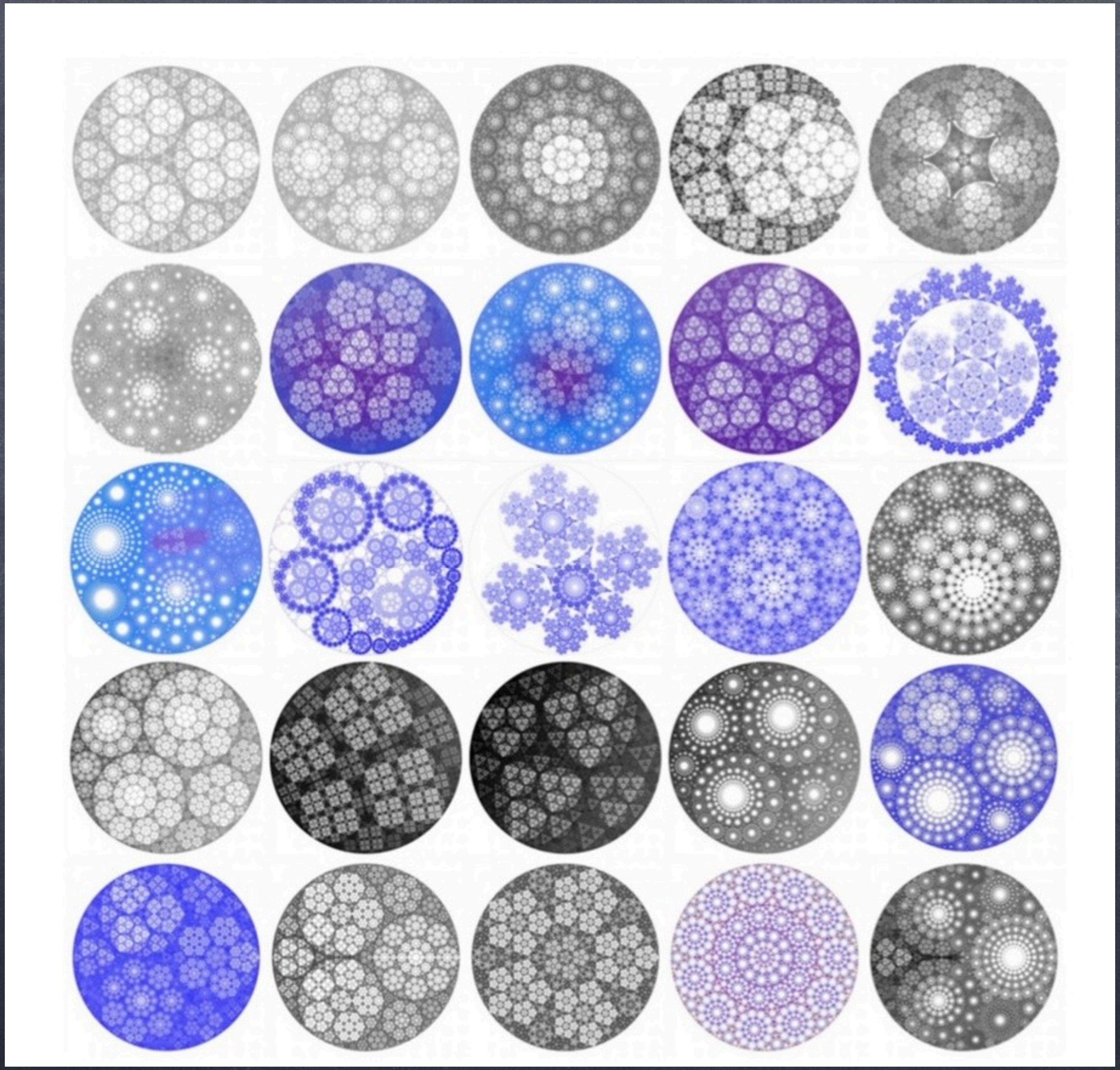



Start with three tangential circles in $\mathbb{C}$.

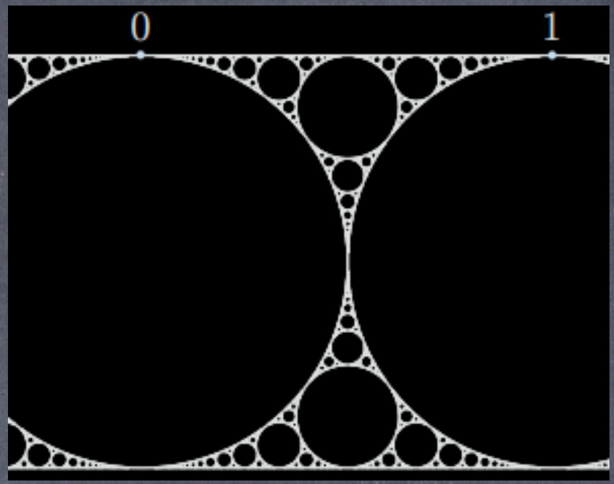
Add two circles tangent to them. Repeat.



# Kleinian Circle Packing : = limit set of Kleinian gp



# III. Kleinian circle packings

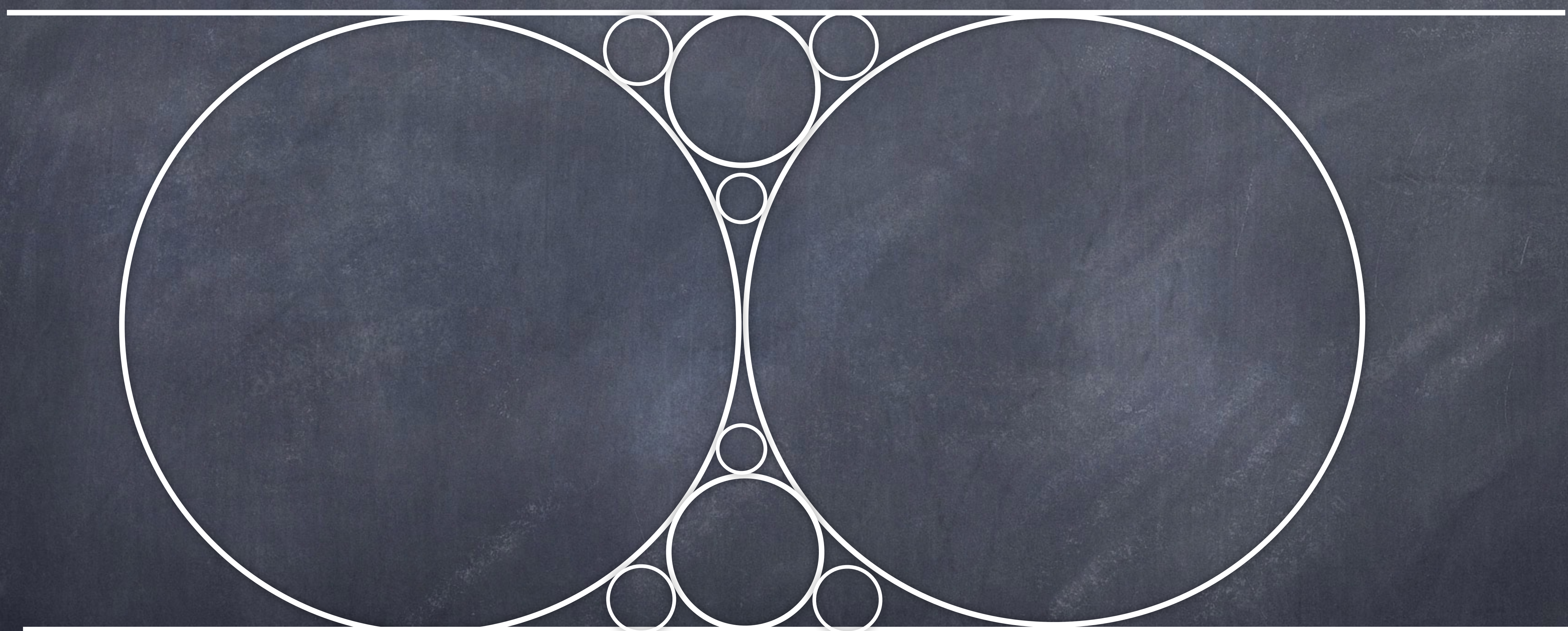


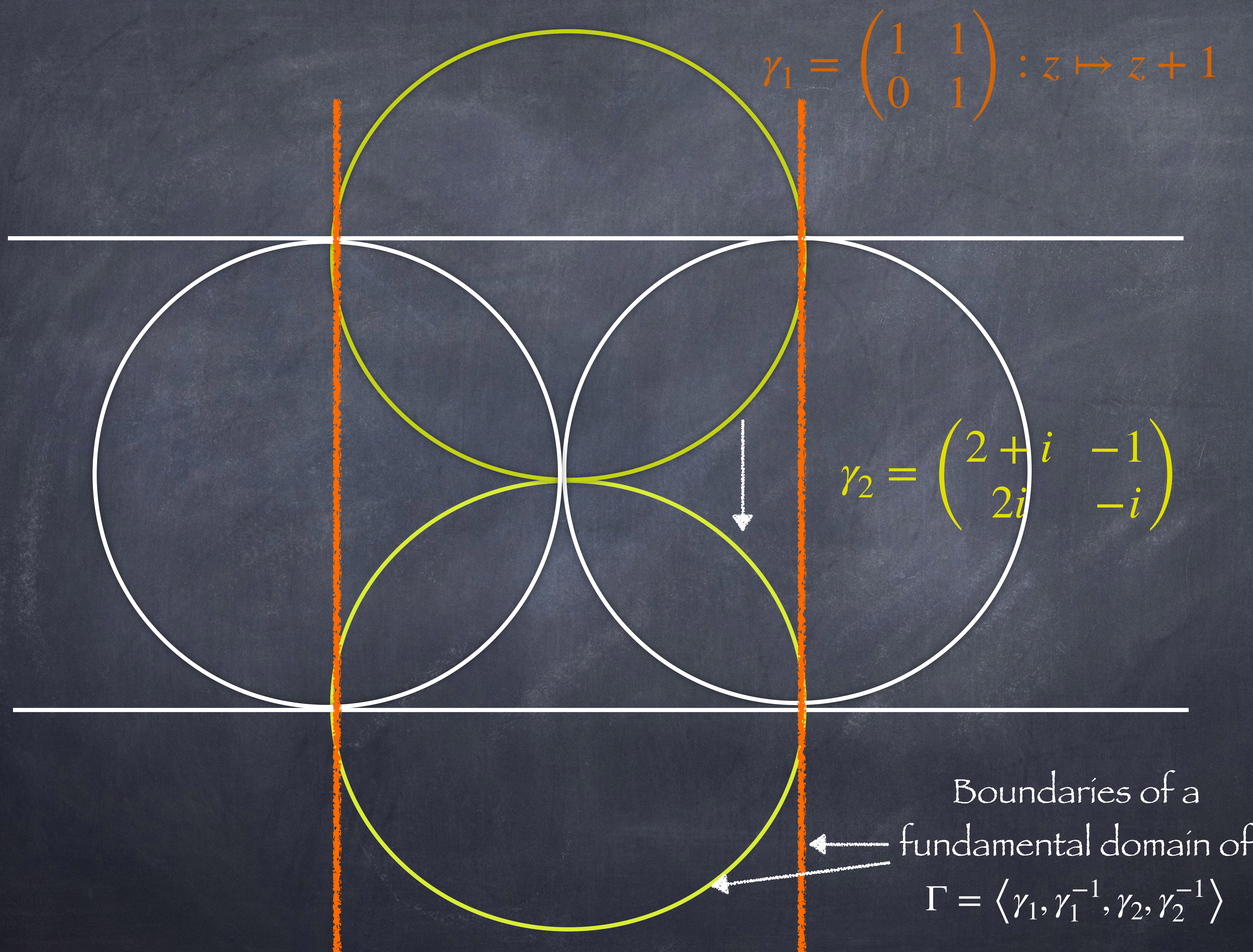

Kleinian Circle Packing : = limit set of Kleinian gp

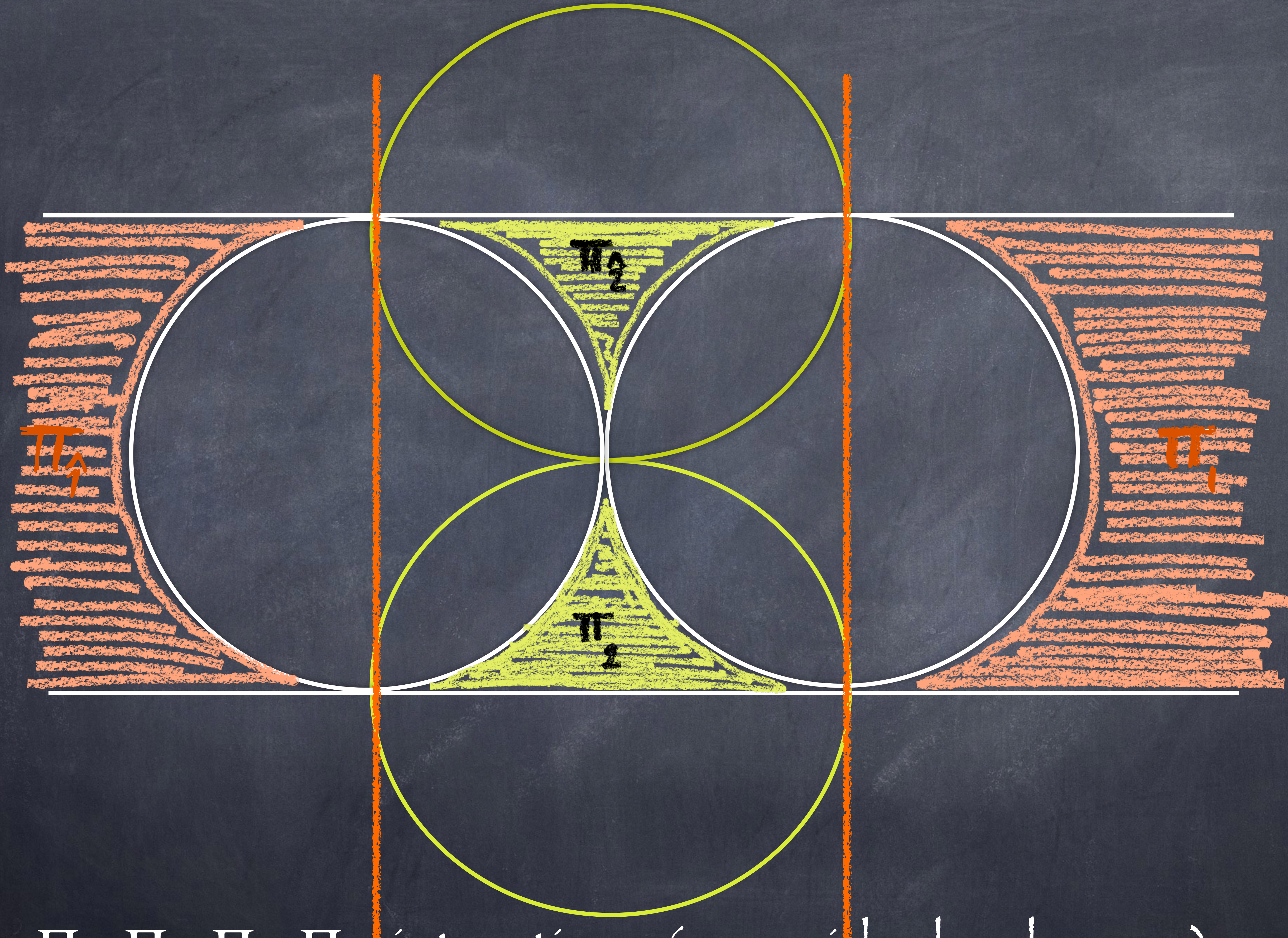
$\Gamma \leq G = PSL_2(\mathbb{C})$ discrete subgroup, assume geometrically finite

Y. Luo-Y. Zhang There exists side pairing maps.

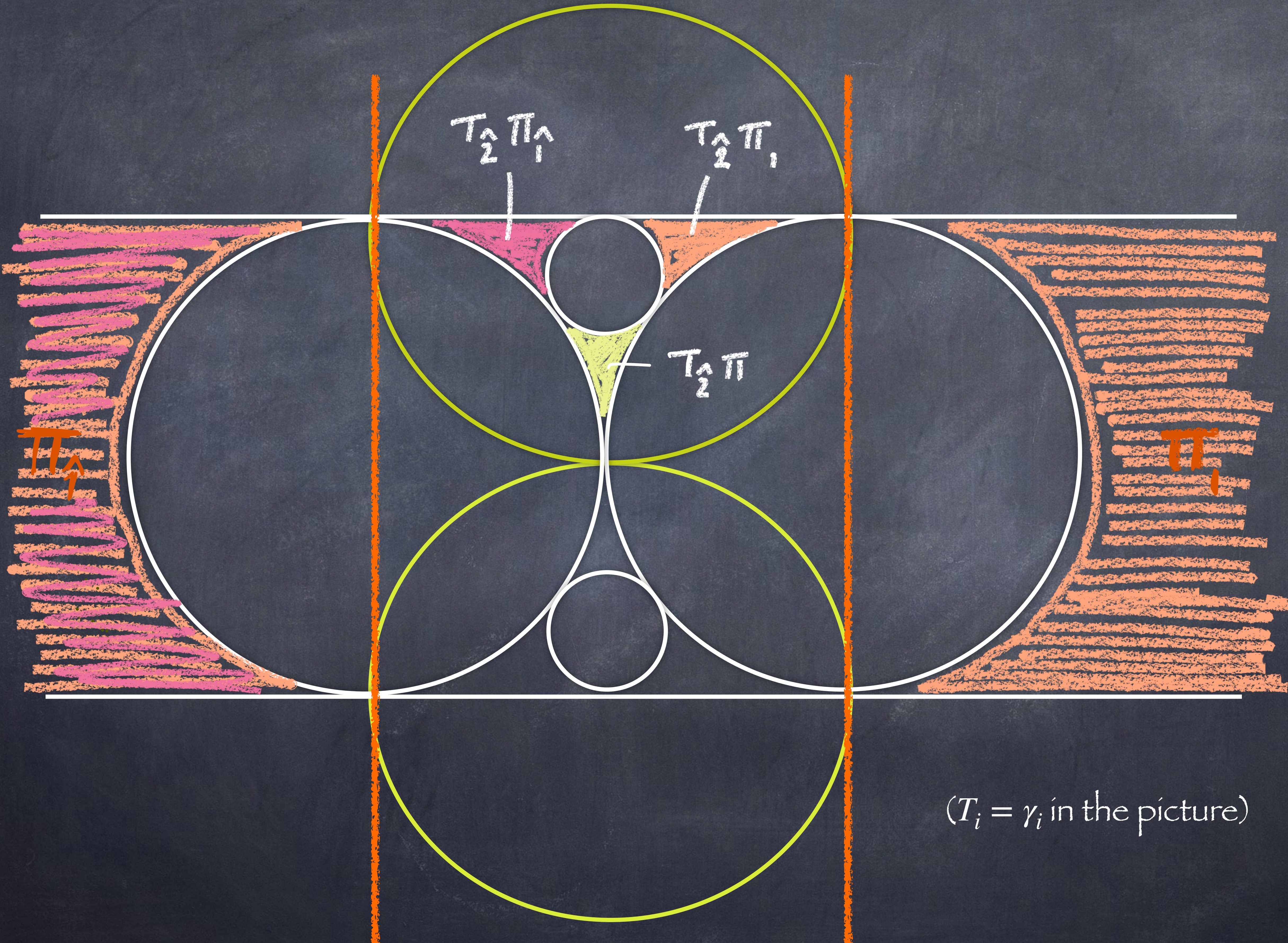
E.g. Side pairing maps for Apollonian circle packing



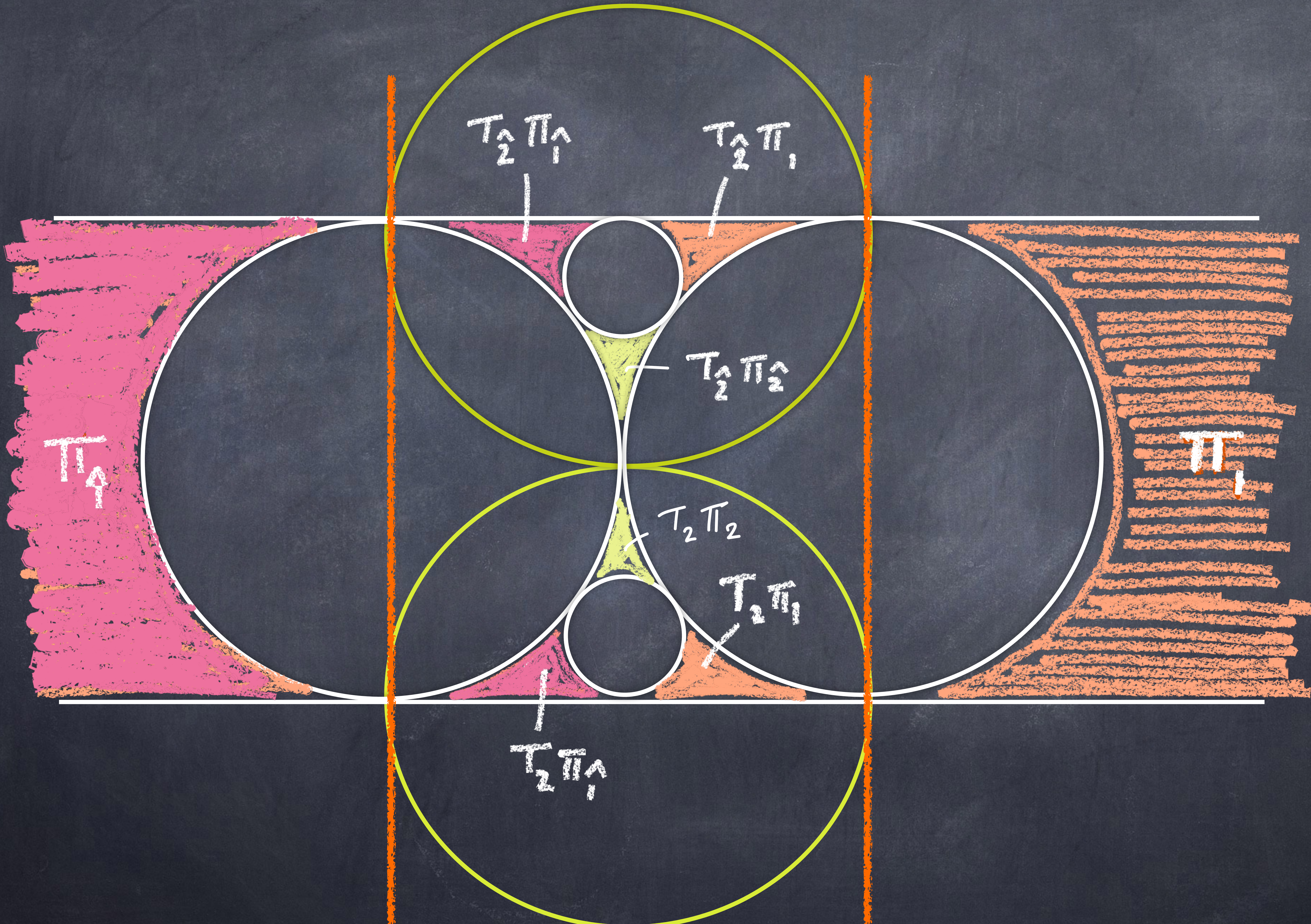

$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z + 1$
$\gamma_2 = \begin{pmatrix} 2+i & -1 \\ 2i & -i \end{pmatrix}$
Boundaries of a fundamental domain of $\Gamma = \langle \gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1} \rangle$




$\Pi_1, \Pi_{\hat{1}}, \Pi_2, \Pi_{\hat{2}}$: interstices (open ideal polygons)




($T_i = \gamma_i$ in the picture)

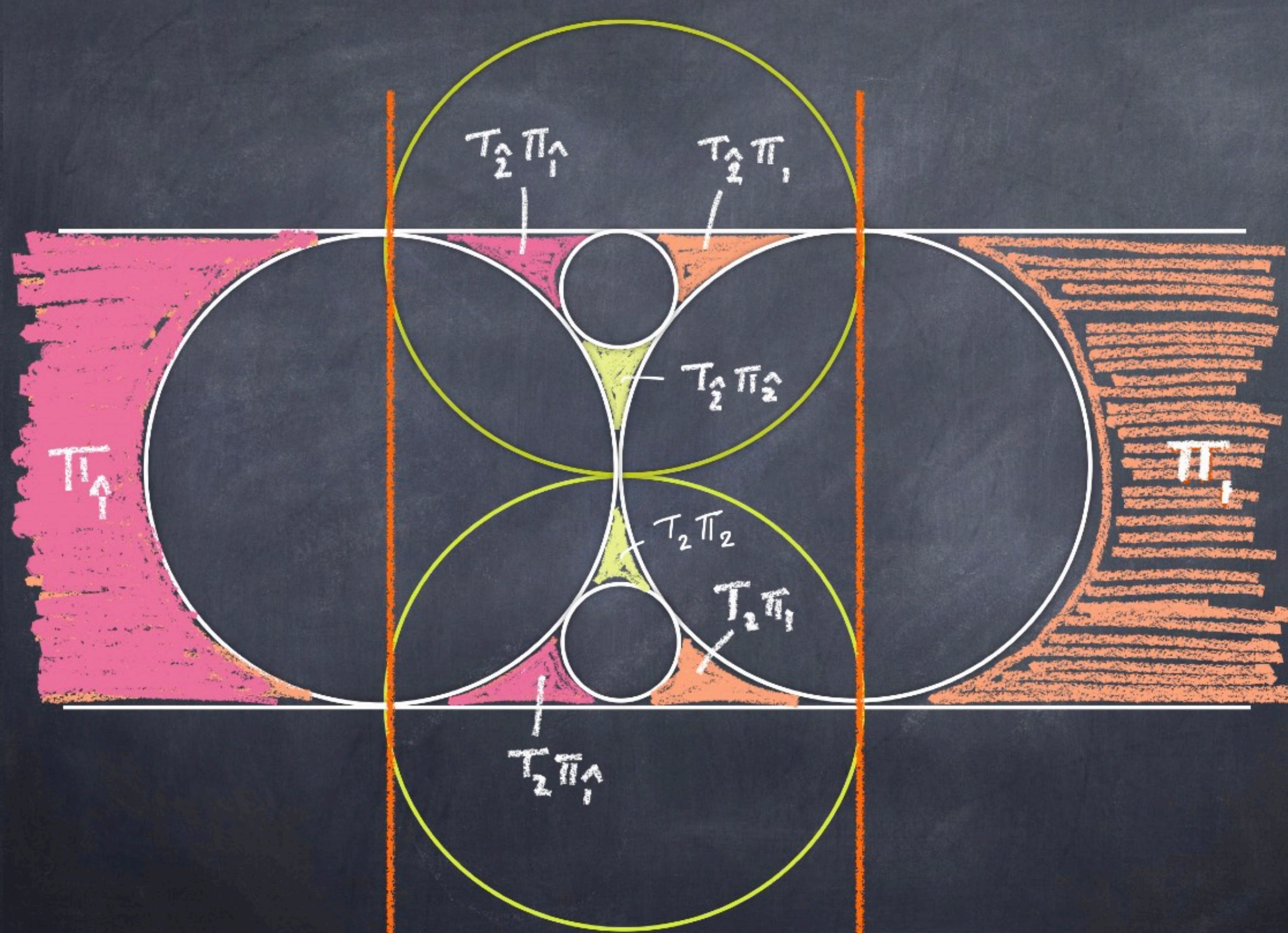


$T_{\hat{2}}\pi_{\hat{1}}$
$T_{\hat{2}}\pi_{1}$
$T_{\hat{2}}\pi_{\hat{2}}$
$\pi_{\hat{1}}$
$\pi_{1}$
$T_{2}\pi_{2}$
$T_{2}\pi_{1}$
$T_{2}\pi_{\hat{1}}$

$\omega$ :words of $1, 2, \hat{1}, \hat{2}$ without backtracking

For $\omega = a_1 a_2 \cdots a_n$, $\gamma_\omega := \gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n}$.

Circle packing $\mathscr{P} = \bigcap_{n \geq 1} \bigcup_{|\omega| = n, i = 1, 2, \hat{1}, \hat{2}} \gamma_\omega \circ \Pi_i$




Suppose that there is a cycle such that non-adjacent vertices in C are non-adjacent in $\mathscr{P}$. Then $\mathbb{C} - \mathscr{C}$ is a union of two ideal polygons $\Pi_i, \Pi_i^c$.

Thm (Luo-Zhang) Suppose there is no rk 2 cusp (in this talk).

Then $\exists 2d$ maps $\gamma_1, \ldots, \gamma_d$ generating $\Gamma$ and $\exists 2d$ disjoint open domains $\Pi_1, \Pi_{\hat{1}}, \ldots, \Pi_d, \Pi_{\hat{d}}$, s.t.

(1) $\Lambda = \bigcup_{k \in \mathscr{A}} \Lambda(\Pi_k)$, where $\mathscr{A} = \{1, \cdots, d, \hat{1}, \cdots, \hat{d}\}$.

(2) $\gamma_k(\overline{\Pi^c_{\hat{k}}}) = \overline{\Pi_k}$.

(3) $\gamma_k(\widehat{\mathbb{C}} - \mathscr{D}_{\hat{k}}) = \overline{\mathscr{D}_k}$, where $\mathscr{D}_k, \mathscr{D}_{\hat{k}}$ are the enlargements of $\Pi_k, \Pi_{\hat{k}}$, resp.

Prop (Park-L.-Zhang) There exists a Bowen-Series map defined by

$$\gamma_k^{-1}(x) \text{ if } x \in \Pi_k .$$

Caution: Bowen-Series map is not necessarily contracting (nor expanding)

Remedy: accelerate along cuspidal words

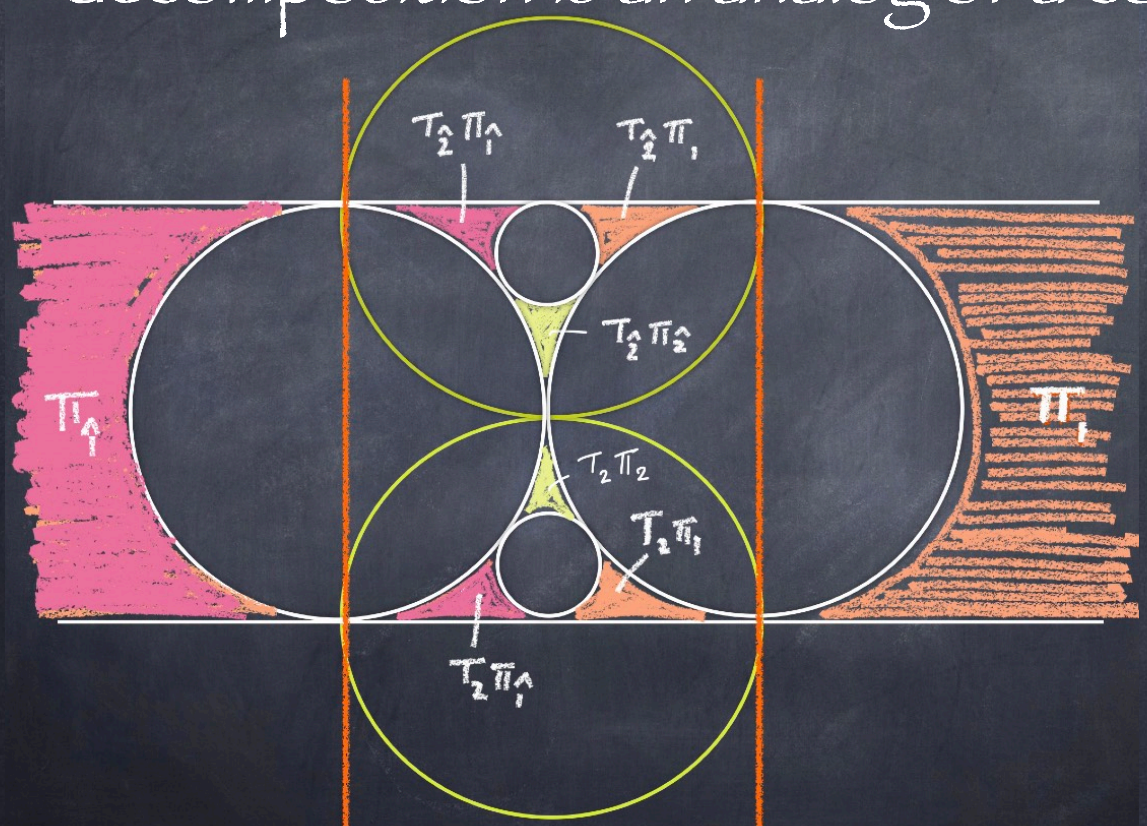
(Words that are extendable to a parabolic word).

Cuspidal decomposition: $\omega = \omega_0 \omega_1 \cdots \omega_m$.

Examples of parabolic words: $\gamma_i^n$, $(\gamma_2 \circ \gamma_1 \circ \gamma_{\hat{2}} \circ \gamma_{\hat{1}})^n$.

$\infty$ $0$

Acceleration of BS map on the sequences of cuspidal decomposition is an analog of a complex CF map.




Even more cceleration of BS map

-> conformal graph-directed Markov system

-> thermodynamic formalism available

Thm (L.-Park-Zhang) (Good approximation)

For any nonnegative integer $r$ with $|W_r| > d$, we have

$$\frac{1}{|W_r| + 2\mu + d} \leq D(\zeta_r)^2 \cdot |\alpha - \zeta_r| \leq \frac{1}{|W_r| - d}.$$

Moreover, $\exists \epsilon_0 > 0$ depending only on $\Pi_j$, so that for any $\gamma \in \Gamma$ and pt $p \in V(\Pi_j)$ with $D(\gamma \cdot p) \neq 0$, we have that

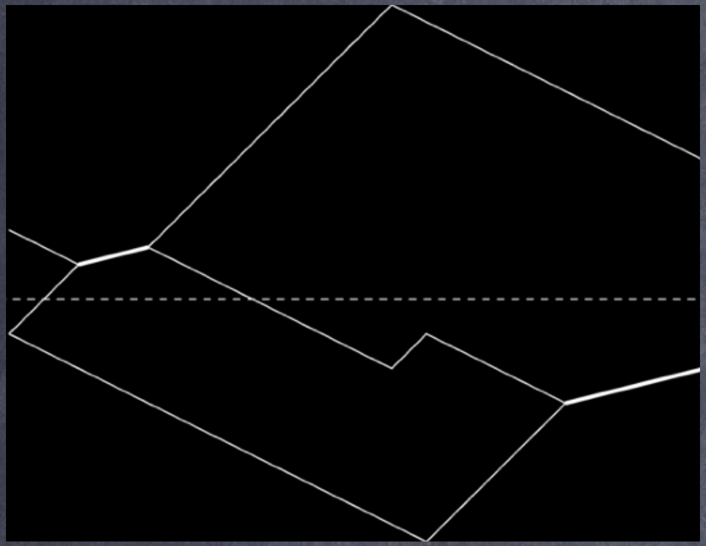
$$D(\gamma \cdot p)^2 |\alpha - \gamma \cdot p| < \epsilon_0 \;\Rightarrow\; \gamma \cdot p = \zeta_r$$

$r \in \mathbb{Z}_{\geq 0}$ with $|W_r| > 0$.

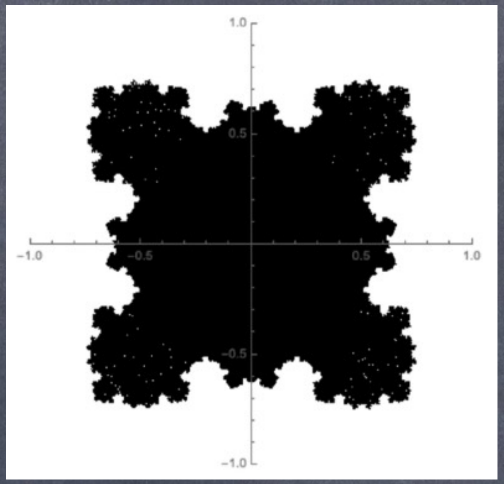
L. Marchese $\dim_H(Bad_\varepsilon) = 1 - C\varepsilon + o(\varepsilon)$ for Fuchsian gps of finite covolume.

We expect to obtain a similar result (for infinite covolume).

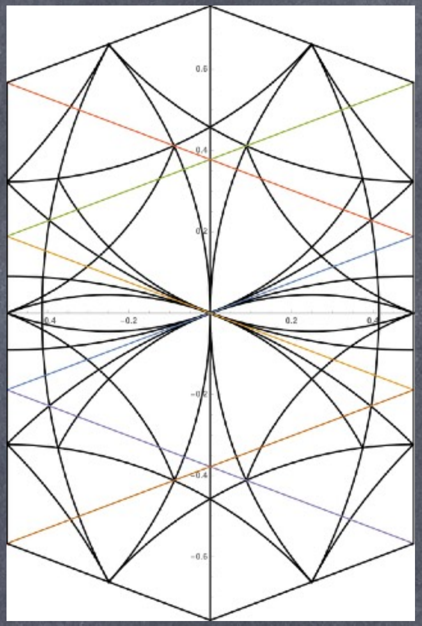
# Image credits



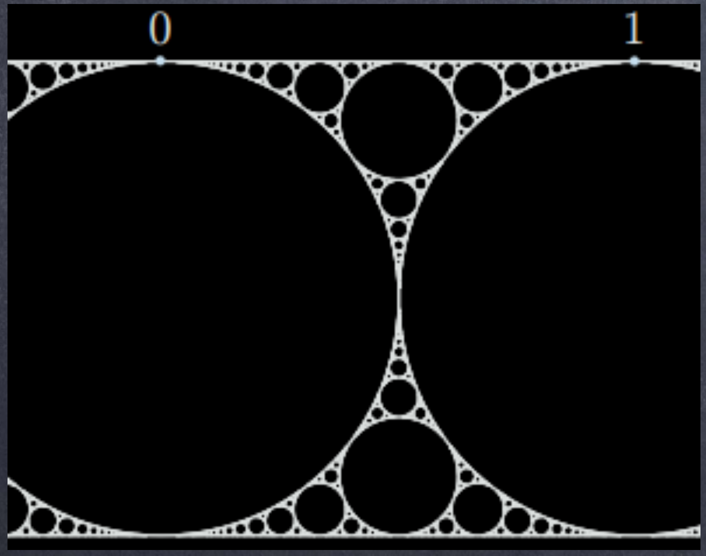
Templates in PGoN:Das-Fishman-Simmon-Urbański (2024)



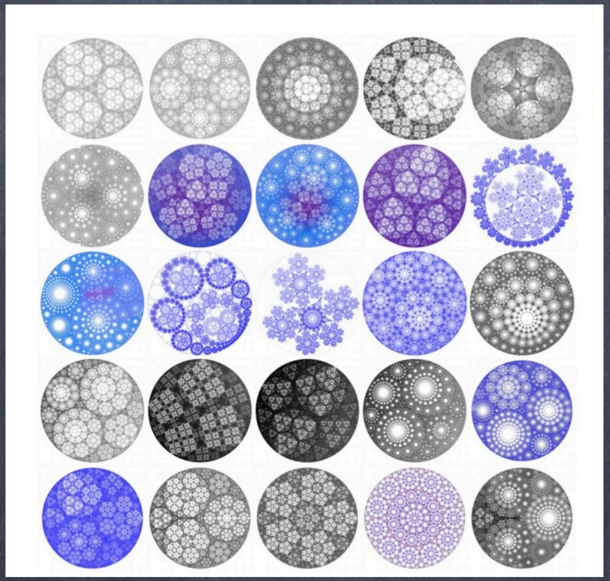
Domain of dual system of C-Gauss map: by Ei-Nakada-Natsui (2023)



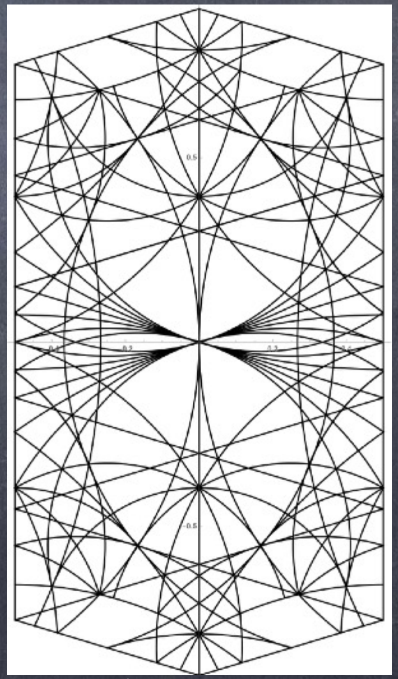
d=7 : Ei-Nakada-Natsui (2023)




Apollonian circle packing: Y. Zhang (arxiv:2111.10277)



Kleinian circle packings:Katherine Stange



d=11 : Ei-Nakada-Natsui (2023)