Spectrality of Moran measures with alternating-sign contraction ratios

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Question

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If this is the case, we call such μ a spectral measure; Ω a spectral set if μ is the Lebesgue measure; Λ the spectrum.

Motivation

Let $\Omega \subset \mathbb{R}^n$ be an open region and let f be such that $\partial_i f, \partial_i\partial_j f$ exist and continuous, then $\partial_i\partial_j f = \partial_j\partial_i f$. There are examples that ∂_i and ∂_j do not commute.

Segal (1958): Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open connected region. Can we extend $\partial_1, \ldots, \partial_n$ on $C^\infty(\Omega)$ to be commutative self-adjoint operators on $L^2(\Omega)$?

Fuglede (1974): If $\Omega \subset \mathbb{R}^n$ is a Nikodym region. Then Segal's problem is true if and only if $L^2(\Omega)$ admits an exponential orthonormal basis $\{e^{2\pi i<\lambda,x>} : \lambda \in \Lambda\}$.

Motivation

Fuglede Conjecture (1974): For bounded measurable set $\Omega \subset \mathbb{R}^n$. Ω is a spectral set if and only if Ω is a tile of $\mathbb{R}^n$ (i.e. $\exists\ \Gamma \subset \mathbb{R}^n$ s.t. $\Omega + \Gamma = \mathbb{R}^n$ with measure disjoint union).

Tao (2004): gave an example of spectral set that is not a tile on $\mathbb{R}^n$ where $n \geq 5$. (improved to $n \geq 3$ later)

Kolountzakis and Matolcsi (2006): disproved the other direction.

It remains open for $n \leq 2$.

Lev and Matolcsi (Acta Math. 2022): proved that Fuglede Conjecture for convex domains is true in all dimensions.

Motivation

A more intriguing development lies in the study of self-similar measures, where the spectral property has only recently begun to be investigated.

For the self-similar measure, a typical example is the Cantor measure: $\mu_\rho, 0 < \rho < 1/2$ where

$$\mu_\rho(\cdot) = \frac{1}{2}\mu_\rho(\rho^{-1}(\cdot) - 1) + \frac{1}{2}\mu_\rho(\rho^{-1}(\cdot) + 1).$$

Question: Is μ_ρ a spectral measure?

Motivation

In a seminal paper, Jorgensen and Pedersen (1998) first showed that: $\mu_{1/n}$ is a spectral measure if and only if n is even (that is, $2 \mid n$.)

This generates a lot of interesting and far reaching questions. Please refer to many works of Jorgenson, Pederson, Laba, Strichartz, Tao, Dutkay, Wang, Lau, Dai, He, Li, An, Fu, Lai etc.

Self-similar measure

Let $0 < \rho < 1$, $N \geq 2$ and $D_N = \{0, 1, \ldots, N-1\}$ be a consecutive digit set. The IFS $\{\phi\}_{d \in D_N}$ is defined by

$$\phi_d(x) = \rho(x + d), \quad d \in D_N.$$

It is well known that there exists a unique Borel probability measure μ_ρ such that

$$\mu_\rho = \frac{1}{N} \sum_{d \in D_N} \mu_\rho \circ \phi_d^{-1}.$$

Such μ_ρ is called a self-similar measure.

Theorem (Dai-He-Lau, Adv. Math. 2014)

μ_ρ is a spectral measure if and only if $p := \rho^{-1}$ is an integer and $N \mid p$.

Self-similar measure with alternating-sign

Recently, Wu first considered the self-similar measure with alternating-sign contraction ratios as follows:
Let $0 < \rho < 1, N \geq 1$ and $D_{2N} = \{0, 1, \ldots, 2N-1\}$ be a consecutive digit set.
Let $\mu_{\pm\rho}$ be the self-similar measure generated by the IFS $\{\phi_d\}_{d \in D_{2N}}$, where

$$\phi_d(x) = (-1)^d \rho(x+d), \quad d \in D_{2N}.$$

Theorem (Wu, Adv. Math. 2024)

$\mu_{\pm\rho}$ is a spectral measure if and only if $p := \rho^{-1}$ is an integer and $2N \mid p$.

Moran measure

Moran measures, which generalize self-similar measures, are characterized by intricate infinite product structures. The spectral analysis of Moran measures was initiated by Strichartz in 2000.

Let $b_n \geq 2$ be integers and $\mathcal{D}_n = \{0, 1, \ldots, N_n - 1\}$ with $N_n \geq 2$.

Theorem (An-Li-Zhang, PJM, 2025)

The sequence of discrete measures

$$\mu_n = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \delta_{(b_1 b_2 \cdots b_n)^{-1}\mathcal{D}_n}$$

converges weakly to a Borel probability measure μ if and only if

$$\sum_{n=1}^{\infty} \frac{N_n}{b_1 b_2 \cdots b_n} < \infty.$$

In this case, μ is called a Moran meausure.

Moran measure

Theorem (An-He, JFA, 2014, Deng-Li, JMAA, 2022; An-Li-Zhang, PJM, 2025)

With the same assumption, the following are equivalent:

1. *The Moran measure μ is a spectral measure.*
2. *There exists a Borel probability measure ν such that $\mu * \nu = Leb_{[0, N_1/b_1]}$.*
3. $N_n \mid b_n$ *for each* $n \geq 2$.

Moran measure with alternating-sign

Let $\{b_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty}$ be two sequences of integers with all $b_k \geq n_k \geq 2$.

Definition (Moran IFS)

Let $\Phi_k = \{\phi_{k,i}\}_{i=0}^{n_k-1}$ be a family of contracting similitudes on $\mathbb{R}$ for each $k \geq 1$, where

$$\phi_{k,i}(x) = (-1)^i b_k^{-1}(x+i), \quad x \in \mathbb{R} \quad (i = 0, 1, \dots, n_k - 1).$$

We call $\{\Phi_k\}_{k=1}^{\infty}$ a Moran iterated function system (IFS) with alternating-sign contraction ratios.

Moran measure with alternating-sign

Theorem (Luo-Mao-Liu, preprint, 2025)

1. *There exists a unique sequence of nonempty compact sets* $\{E_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}$ *such that*

$$E_k = \Phi_k(E_{k+1}) := \bigcup_{i=0}^{n_k-1} \phi_{k,i}(E_{k+1}), \quad k \geq 1.$$

2. *There exists a unique sequence of probability measures* $\{\mu_k\}_{k=1}^{\infty}$ *satisfying*

$$\mu_k = \frac{1}{n_k}\mu_{k+1} \circ \Phi_k^{-1} := \sum_{i=0}^{n_k-1} \frac{1}{n_k}\mu_{k+1} \circ \phi_{k,i}^{-1}, \quad k \geq 1,$$

where each μ_k *is supported on* E_k.

Moran measure with alternating-sign

The first measure $\mu := \mu_1$ is called a *Moran measure with alternating-sign contraction ratios*, which will be our research object. Our first main result is

Theorem (Luo-Mao-Liu, preprint, 2025)

Suppose that the sequence $\{b_k\}_{k=1}^{\infty}$ is bounded and $4 \mid n_k$ for all $k \geq 1$. Then μ is a spectral measure if and only if

$$n_2 \mid 2b_2 \quad \text{and} \quad n_k \mid b_k \text{ for } k \geq 3.$$

This result extends the previous frameworks proposed by An-He 2014, Deng-Li 2022, and Wu 2024, providing a generalized criterion for such measures.

Moran measure with alternating-sign

Remark: The assumption that $4 \mid n_k$ is crucial; without it, we cannot obtain the necessary condition of the spectrality of μ.

Theorem (Luo-Mao-Liu, preprint, 2025)

Suppose that the sequence $\{b_k\}_{k=1}^{\infty}$ is bounded and $2 \mid n_k$ for all $k \geq 1$. Then μ is a spectral measure if

$$2 \mid b_2, \quad n_2 \mid 2b_2 \quad \text{and} \quad n_k \mid b_k \text{ for } k \geq 3.$$

Moran measure with alternating-sign

The primary challenges in establishing the main theorems arise from two fundamental aspects:

1. Unlike classical self-similar measures, the Moran measure μ lacks a convolution structure, which prevents direct analysis via the framework of infinite convolution measures.
2. To address the necessity part, we develop a multi-stage spectral decomposition, which enables us to extract hierarchical arithmetic constraints by recursively partitioning the spectrum of the measure.

Our proof strategy

By assumption, n_k's are all even numbers, we let $n_k = 2p_k$ where $p_k \in \mathbb{N}$ for $k \geq 1$. The measures $\{\mu_k\}_{k=1}^{\infty}$ can be reformulated as

$$\mu_k = \sum_{j=0}^{2p_k-1} \frac{1}{2p_k} \mu_{k+1} \circ \phi_{k,j}^{-1}, \quad k \geq 1.$$

Lemma (First key lemma)

Let $\mu := \mu_1$ be the Moran measure defined by the above, then its Fourier transform is

$$\widehat{\mu}(t) = e^{-b_1^{-1}\pi t i} \prod_{k=1}^{\infty} f_k\left((b_k \cdots b_1)^{-1} t\right), \quad t \in \mathbb{R},$$

where $f_k(t) = \frac{1}{p_k} \sum_{j=0}^{p_k-1} \cos(4j + 1 - b_{k+1}^{-1})\pi t.$

Our proof strategy

Let $D_n = \{0, 1, \ldots, n-1\}$ for $n \geq 1$, and let

$$\mathcal{D}_k = D_{p_k} \oplus (p_k - \frac{1 + b_{k+1}^{-1}}{2}) D_2.$$

Define an infinite convolution measure $\nu := \nu_{\{b_k\},\{\mathcal{D}_k\}}$ by

$$\nu = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1 \cdots b_k)^{-1}\mathcal{D}_k} * \cdots .$$

This measure ν differs significantly from that in the theorem of An–Li–Zhang 2025, since the digit set $\mathcal{D}_k$ is no longer consecutive.

Lemma (Second key lemma)

The Moran measure $\mu := \mu_1$ is a spectral measure if and only if ν is a spectral measure.

Thank you for your attention !