The Brjuno and Wilton functions

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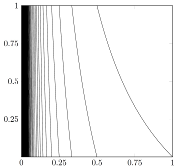
Continued fractions and the Gauss map

The Gauss map $A_1 : (0,1) \mapsto [0,1]$ is

$$A_1(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right].$$

$$x = a_0 + x_0 = a_0 + \frac{1}{a_1 + x_1} = \ldots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + x_n}}},$$



and we will write $x = [a_0; a_1, \ldots, a_n, \ldots]$. The *nth-convergent* of x is

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}.$$

Diophantine numbers and Brjuno numbers

- Let $\beta_n = \prod_{i=0}^{n} x_i = (-1)^n (q_n x - p_n)$ for $n \geq 0$, and $\beta_{-1} = 1$.
- *Diophantine numbers*: $q_{n+1} = \mathcal{O}(q_n^{1+\tau})$ for some $\tau \geq 0$, i.e. $a_{n+1} = \mathcal{O}(q_n^{\tau})$.
- *Brjuno numbers*: $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty$.
- All diophantine numbers are Brjuno numbers but also "many" Liouville numbers are Brjuno numbers: for example $\sum_{n\geq 1} 10^{-n!}$ is a Brjuno number.
- Brjuno numbers where introduced by Thomas Cherry in 1964. Cherry conjectured the Brjuno condition to be sufficient for the existence of Siegel disks in holomorphic dynamics. This was proved by Brjuno shortly afterwards .
- In 1988 Jean-Christophe Yoccoz proved that *the Brjuno conditon is not only sufficient but also necessary.*

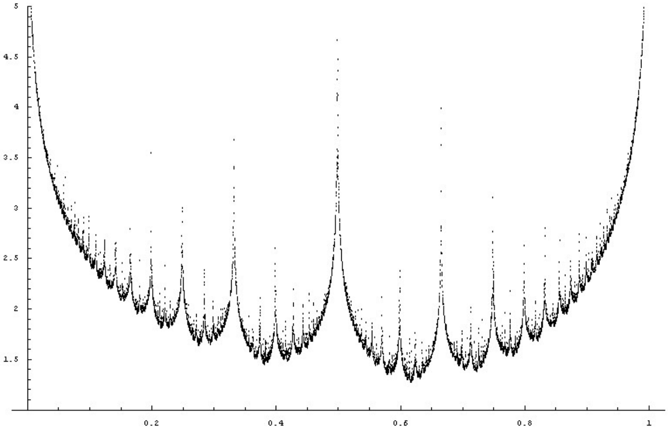
Brjuno numbers and the Brjuno function

- The *Brjuno function* $B : \mathbb{R} \setminus \mathbb{Q} \to (0, +\infty]$,

$$B(x) := \sum_{n=0}^{\infty} \beta_{n-1} \log x_n^{-1} < +\infty .$$

  was introduced by Yoccoz [Yoc88, Yoc95] around 1988.
- x is a *Brjuno number* if and only if $B(x) < +\infty$. Indeed there exists $C > 0$ such that $\left| B(x) - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \right| \leq C$.
- The *Hölder continuity* conjecture (aka known as MMY conjecture) [MMY97] in holomorphic dynamics is a 30–years old open problem stating that the log–size of quadratic Siegel disks is given by the Brjuno function *up to a* 1/2*–Hölder continuous correction*.
- In 2006 Buff and Chéritat [BC06] proved that the correction is continuous. Later Chéritat proved that the Hölder exponent cannot exceed $\frac{1}{2}$.
- In 2015 Chéritat and Cheraghi [CC15], using the renormalization invariant class of Inou and Shishikura, proved of the Hölder interpolation conjecture for rotation numbers of high type. However they form a set of zero measure.

The Brjuno function



Brjuno functions as cohomology classes

- The Brjuno function satifies a functional equation under the action of $\mathrm{PGL}(2,\mathbb{Z})$:

$$B(x) = B(x+1) \quad \forall x \in \mathbb{R} \setminus \mathbb{Q}$$

$$B(x) = -\log x + xB\left(\frac{1}{x}\right), \quad \forall x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$$

- Thus the set of Brjuno numbers is $\mathrm{PGL}(2,\mathbb{Z})$ invariant.
- Quadratic irrationals have an eventually periodic continued fraction expansion, thus B is known *exactly* on a countable but dense set of irrationals.
- If one restricts B to $x \in (0,1)$ then the above relations correspond to a *twisted cohomological equation* corresponding to the dynamics of the Gauss map:

$$B(x) = -\log x + xB \circ A_1(x), \quad \forall x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$$

The Wilton function[Wil33]

- Let $W(x) = \sum_{n=0}^{\infty}(-1)^n \beta_{n-1}(x) \log x_n^{-1}$. It's the alternate signs version of the sum defining the Brjuno function: $B(x) = \sum_{n=0}^{\infty} \beta_{n-1}(x) \log x_n^{-1}$.
- W converges if and only if $\left|\sum_{n=0}^{\infty}(-1)^n \frac{\log(q_{n+1}(x))}{q_n(x)}\right| < \infty$.
- The points of convergence are called *Wilton numbers*: in 1933 Wilton proved that they are the points where the trigonometric series

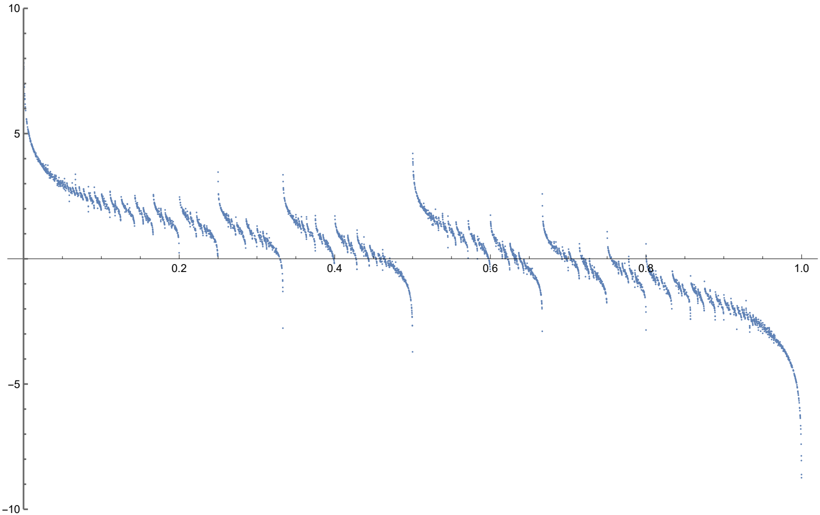
$$\sum_{n=0}^{\infty} \frac{d(n)}{n} \sin(2\pi n x)$$

converges. Here $d(n)$ is the number of divisors of n: $d(n) = \sum_{k|n, k\geq 1} 1$.
- The function W is 1–periodic and when $x \in [0,1)$ is a Wilton number:

$$W(x) = \log\left(\frac{1}{x}\right) - xW\left(\frac{1}{x}\right), \text{whereas } B(x) = \log\left(\frac{1}{x}\right) + xB\left(\frac{1}{x}\right)$$

In [LMPS23] a complexification of the Wilton function is constructed using the same methods developed in [MMY01].

The Wilton function



John Raymond Wilton's 1933 theorem

An approximate functional equation with applications to a problem of Diophantine approximation.

By *J. R. Wilton* in Adelaide (South Australia).

1. In recent papers, published at the same time, Chowla [1]) and Walfisz [2]) have proved a number of results as to the order of magnitude of $\sum_{n \leqq \omega} d(n) \cos 2\pi n x$ and kindred sums, where $d(n)$ is the number of divisors of the positive integer n, and x is (except in one case) irrational. In the present communication I obtain most of their theorems and some new results by a systematic application of an approximate functional equation for the sum considered; the method is that originated by Hardy and Littlewood nearly twenty years ago [3]), and employed by Oppenheim [4]) to prove some analogous theorems concerning $\sum_{n \leqq \omega} r(n) e^{i\pi n x}$, and more general sums [5]), where $r(n)$ is the number of ways of expressing n as the sum of two squares.

The condition for the convergence of the series (10) may be stated with precision:

(19$_{\text{III}}$) $\sum_{n \geqq 1} \frac{d(n)}{n} \cos 2\pi n x$ *converges if and only if* $\sum_{r \geqq 0} x x_1 \cdots x_{r-1} \log^2 \frac{1}{x_r}$ *is convergent* [9]);

(20$_{\text{III}}$) $\sum_{n \geqq 1} \frac{d(n)}{n} \sin 2\pi n x$ *converges if and only if* $\sum_{r \geqq 0} (-)^r x x_1 \cdots x_{r-1} \log \frac{1}{x_r}$ *is convergent* [9]).

It will be plain from this summary that Wilton was a fine mathematician, with admirable taste and a natural inclination towards deep and difficult problems. ... his record is genuinely impressive. He might perhaps have made a bigger name if his taste had been less fine, and he had been content to work in fields which offer cheaper rewards. Wilton's obituary, H.S. Carlslaw and G.H. Hardy, J. London Math Soc. January 1945 pp. 58–64

Wilton's theorem and the Brjuno function

7. In the same way as Theorem 3 is derived from Theorem 1, we derive from (2. 2) and (2. 21)

Theorem 4. *If x is irrational, $0 < x < 1$, $\omega > 1 + A$, and m is given by* (6. 1), *then*

$$\sum_{n \leq \omega} \frac{d(n)}{n} \cos 2\pi n x = \mathfrak{F}(x) + o(1) + X_{m-1} \tag{7. 1$_{\text{III}}$}$$

$$+ x x_1 \cdots x_{m-1} \int_1^{\omega_m} \frac{\log t + 2\gamma}{t} \cos 2\pi x_m t \, dt, \tag{7. 11}$$

$$\sum_{n \leq \omega} \frac{d(n)}{n} \sin 2\pi n x = \mathfrak{F}(x) + o(1) + \frac{1}{2}\pi X'_{m-1} \tag{7. 2$_{\text{III}}$}$$

$$+ (-)^m x x_1 \cdots x_{m-1} \log \frac{1}{x_m} \int_0^{2\pi x_m \omega_m} \frac{\sin t}{t} \, dt, \tag{7. 21}$$

where

$$X_m = \frac{1}{2}\left(\log^2 \frac{1}{x} + x \log^2 \frac{1}{x_1} + \cdots + x x_1 \cdots x_{m-1} \log^2 \frac{1}{x_m}\right) \tag{7. 31}$$

$$- (\log 2\pi - \gamma)\left(\log \frac{1}{x} + x \log \frac{1}{x_1} + \cdots + x x_1 \cdots x_{m-1} \log \frac{1}{x_m}\right),$$

$$X'_m = \log \frac{1}{x} - x \log \frac{1}{x_1} + \cdots + (-)^m \, x x_1 \cdots x_{m-1} \log \frac{1}{x_m}. \tag{7. 32}$$

Modular forms and analytic number theory

- Manin [Man11] found that the Brjuno function can be used for the calculation of derivatives of classical Dirichlet series related to cusp forms.
- In 2013 Izabela Petrykiewicz [Pet13, Pet14, Pet17] proved several theorems linking the differentiability of modular integrals to functions of Brjuno type.
- Balazard and Martin study the multiplicative autocorrelation of the fractional part and H. Maier and M.Th. Rassias, S. Bettin and J. Conrey, S. Bettin and S. Drappeau have investigated the relation with cotangent sums and the Dedekind eta function.
- Rivoal and Roques [RR13] proved that *An irrational number x is a Brjuno number if and only if*
$$RR(x) := \sum_{n\geq 1} \frac{1}{n^2} \sin(2\pi n^2 x)\cot(\pi n x) < +\infty.$$
- Indeed the difference $B(x) - RR(x)$ is a continuous function of x [CM23].

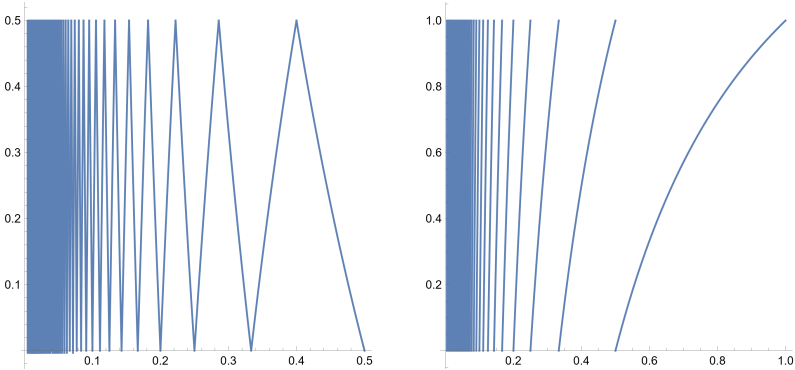
α–continued fractions and their Brjuno functions

- One can embed the Gauss map in a one-parameter family of continued fractions maps [Nak81]: $\alpha \in [0,1]$, $\bar{\alpha} = \max(\alpha, 1-\alpha)$ $A_\alpha : (0,\bar{\alpha}) \mapsto (0,\bar{\alpha})$ is

$$A_\alpha(x) = \left| \left\{ \frac{1}{x} \right\} - \left[\left\{ \frac{1}{x} \right\} + 1 - \alpha \right] \right| .$$

- Only three representatives have full branches: the Gauss map $\alpha = 1$, the *nearest integer continued fraction map* $\alpha = 1/2$ and the *by-excess* (BE) c.f.m. $\alpha = 0$.
- To each A_α one can associate a corresponding Brjuno function $B_\alpha(x) = \sum_{n=0}^{\infty} \prod_{i=0}^{n} x_i \log x_n^{-1}$ where $x_{i+1} = A_\alpha(x_i)$.
- The corresponding set of Brjuno numbers (i.e. $B_\alpha(x) < +\infty$) it has been proved [MMY97, LMNN10] to be independent on $\alpha > 0$.
- $B_1, B_{1/2}$ are cocycles [MMY01] for (a lift of) the standard $\mathrm{PGL}(2,\mathbb{Z})$–action on $\mathbb{R}$.
- **Theorem (Lee, M.) [LM22]** B_0 *is a cocycle under a lift of the standard* $PSL(2,\mathbb{Z})$*–action on* $\mathbb{R} \setminus \mathbb{Q}$.

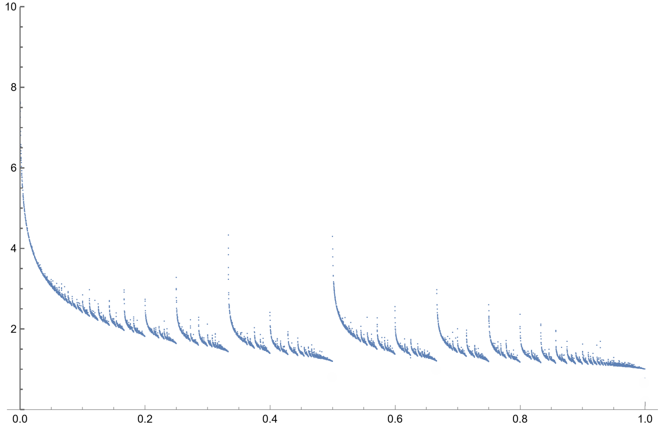
The nearest integer and the by-excess continued fraction maps $A_{1/2}$ and A_0



Hölder continuity of the differences between Brjuno functions

- For the three cases with full branches one can ask if the corresponding Brjuno functions represent the same Hölder cohomology class.
- **Theorem(Moussa, M., Yoccoz) [MMY97]** *The difference between the Brjuno functions associated to the Gauss and to the nearest integer c.f.m. extends to the whole real line* $\mathbb{R}$ *as a periodic* $1/2$*–Hölder continuous function.*
- In 2010 Luzzi, M., Nakada and Natsui [LMNN10] proved that the the difference between the Brjuno function associated to the Gauss map and the even part of the Brjuno function associated to the BE c.f.m. is uniformly bounded (i.e. belongs to L^∞) and conjectured on the basis of numerical evidence that it was indeed 1/2–Hölder continuous function.

The semi–Brjuno function B_0 introduced in [LMNN10]



The odd and the even by-excess Brjuno function

- Let $B_0^{\pm}(x) := \frac{1}{2}(B_0(x) \pm B_0(-x))$. Then there exist constants $C^{\pm}$ such that for all irrational numbers x

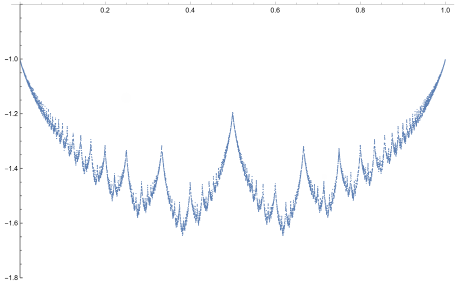
$$|B(x) - 2B_0^{+}(x)| \leq C^{+} \quad \text{and} \quad |W(x) - 2B_0^{-}(x)| \leq C^{-} .$$

- These could be deduced from the work of Wilton, by a modification of his application of Voronoi's summation formula, but one can prove them directly.
- The odd part B^{-} and the even part W^{+} of the Brjuno respectively Wilton functions are *explicit uniformly bounded Hölder continuous functions*
- We thus concentrate on the differences

$$\Delta^{+}(x) := B^{+}(x) - 2B_0^{+}(x) \quad \text{and} \quad \Delta^{-}(x) := W^{-}(x) - 2B_0^{-}(x).$$

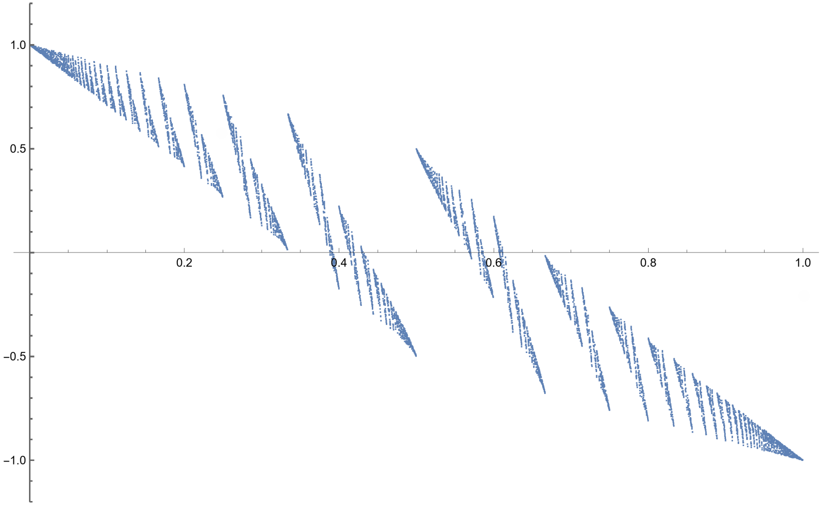
- Numerical computations show that $\Delta^{\pm}$ behave widely differently.

Hölder continuity of $x \mapsto \Delta^+(x)$



Theorem(Seul Bee Lee, M.) [LM22]: $x \mapsto \Delta^+(x)$ *extends to the whole real line* $\mathbb{R}$ *as a periodic* $1/2$*–Hölder continuous function.*

The difference $x \mapsto \Delta^{-}(x) = W^{-}(x) - 2B_0^{-}(x)$

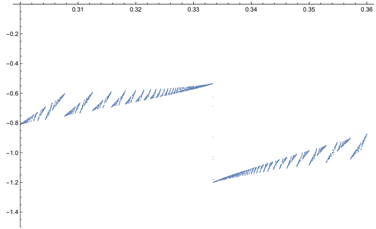


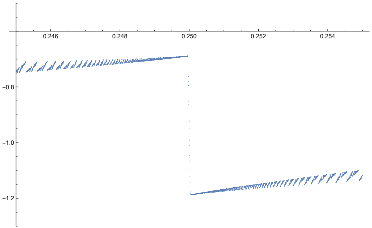
Regularity of $x \mapsto \Delta^-(x)$

- The plot of $\Delta^-(x)$ seems to indicate that $\Delta^-(x)$ is discontinuous at each rational with a jump of order $2/q$ for $x = p/q$.
- In that respect, it is reminiscent of the 'popcorn function'

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ with } (p,q) = 1; \\ 0 & \text{if } x \text{ is irrational;} \end{cases}$$

- In [MMY01] it is proved that the *harmonic conjugate* of the Brjuno funciton is also continuous at all irrational numbers with a decreasing jump of π/q for $x = p/q$.
- **Theorem (Burrin, Lee, M[BLM25])** *The 1-periodic function $x \mapsto \Delta^-(x)$ is continuous at irrationals. At each rational $x = p/q$ it has an increasing jump of $2/q$. In particular Δ^- is Riemann integrable and nowhere differentiable.*





The ingredients of the proof use:

- The automorphic Koopman operator $(Tf)(x) := xf \circ A(x)$, $x \in (0,1)$ and its spectral properties
- The regularity lemmas: the action of T on Hölder continuous functions
- The functional equation for W^- and the propagation of the jumps

The automorphic Koopman operator and L^p regularity

- Let $\nu \geq 0$. We introduce the "automorphic Koopman operator" $(T_\nu f)(x) = x^\nu f \circ A_1$, $x \in (0,1)$, where A_1 is the Gauss map.
- The Brjuno and the Wilton function verify

$$(1 - T_1)B_1 = -\log x\,, \quad (1 + T_1)W(x) = -\log x\,,$$

- Using the absolutely continuous invariant probability measure $\frac{dx}{(1+x)\log 2}$ preserved by A_1 one finds that T_ν has spectral radius bounded by g^ν where $g = \frac{\sqrt{5}-1}{2}$.
- The operators $(1 \pm T_\nu)$ are therefore invertible and their inverse are given by the norm convergent series $\sum_{k\geq 0}(\pm 1)^k T_\nu^k$.

The action on Hölder functions

- The operator T acts on even periodic functions.
- At the level of Hölder continuous functions a new phenomenon appears: there is a non-trivial cohomology class at the level of 1/2–Hölder continuous functions.
- Let f be a continuous function and let B_f denote the solution of $(1-T)B_f = f$.
- **Theorem** The exponent 1/2 plays here a crucial role: if η denotes the Hölder exponent of f then:
  - if $\eta > 1/2$ then B_f is 1/2–Hölder continuous;
  - if $\eta < 1/2$ then B_f is also η–Hölder continuous;
  - if $\eta = 1/2$ then B_f admits $x^{1/2} \log x$ as continuity modulus.

W^- and the propagation of the jumps

- The proof proceeds by identifying the functional equation for the odd 1–periodic function Δ^-, i.e., determining the function $f(x) = \Delta^-(x) + x\Delta^-(1/x)$ for $x \in (0, \frac{1}{2}) \setminus \mathbb{Q}$.
- The function f has an explicit formula: if $x \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ and $n = \lfloor \frac{1}{x} \rfloor$

$$f(x) = -[W^+(x) + xW^+(A_1(x))] - x\log x - xA_1(x)\log(xA_1(x)) - x\sum_{j=1}^{n-1}\log(1-jx)\,.$$

- f extends to an odd 1–periodic function continuous for every $x \in \mathbb{R} \setminus \mathbb{Q}$, with

$$\lim_{x\to 0^+} f(x) = 1, \qquad \lim_{x\to \frac{1}{2}^-} f(x) = 0.$$

  It therefore has a jump of 2 at $x = 0$ but it is continuous at $x = 1/2$.
- We deduce an L^∞ norm convergent series expansion for Δ^-, similarly to those of B and W.
- To show the series thus obtained converges uniformly to a continuous function on $(0, \frac{1}{2}) \setminus \mathbb{Q}$ with the predicted jumps at each rational, we study the regularity of f on the domain of each branch of the iterates of the nearest-integer map $A_{1/2}$.

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