Rotational beta expansions and Schmidt games

(Joint work with Jonathan Caalim and Hajime Kaneko)

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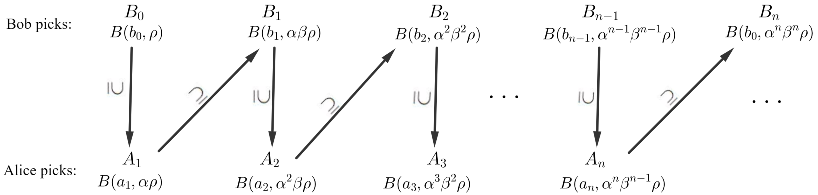
- We have two players: Alice and Bob
- Set up: Alice chooses $S \subseteq X$ which we call the **"target set"**. Bob picks an initial radius $\rho > 0$ and a starting closed ball $B_0 = B(b_0, \rho) \subseteq X$.

Mechanics of Schmidt's Game



The sequence of closed balls converges to a singleton containing the **outcome** ω. Alice wins if $\omega \in S$. Otherwise, Bob wins.

Example

Let $X = \mathbb{R}$ and $S = [0, 1)$. Let $(\alpha, \beta, \rho) = (1/2, 1/3, 1)$.

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$$[-1, 1] \xrightarrow[1/2]{} [-\tfrac{1}{2}, \tfrac{1}{2}] \xrightarrow{1/3} [-\tfrac{1}{6}, \tfrac{1}{6}] \to [-\tfrac{1}{12}, \tfrac{1}{12}] \to [-\tfrac{1}{36}, \tfrac{1}{36}] \to \cdots$$

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Then $\omega = 0 \in S$. Therefore, Alice wins the game.

Winning Sets

Given $\alpha, \beta \in (0, 1)$ and target set $S \subseteq X$, the set S is said to be

- (α, β, ρ)**-winning** ((α, β, ρ)**-losing**) if Alice (Bob) has a winning strategy to win the game.

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Fact: S is winning $\implies$ S is dense. In particular, S has full Hausdorff dimension of $X = \mathbb{R}^m$.

On Numeration Systems

Schmidt's Games were used to study "denseness" of sets related to numeration systems.

Theorem (Zanger-Tishler and Kalia, 2012)

Given $1 < b \in \mathbb{N}$, the set of numbers with 0 in its b-ary expansion is

- (α, β)*-winning if* $\log_b(\alpha\beta) \notin \mathbb{Q}$ *and* $\beta > f_b(\alpha)$ *for some quantity* $f_b(\alpha)$.
- (α, β)*-losing if* $\alpha\beta = \frac{1}{b}$ *and* $\alpha \geq \frac{2}{b}$

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Goal: Find an analogue of the results of Zanger-Tishler and Kalia for rotational beta expansions.

Rotational beta Expansion

Let $\beta > 1$ and $M \in SO(m)$ ($m \in \mathbb{N}$). Let $\mathcal{L}$ be a lattice on $\mathbb{R}^m$ with fundamental domain $\mathcal{X}$ such that $\mathbb{R}^m$ can be partitioned as

$$\mathbb{R}^m = \bigcup_{d \in \mathcal{L}} \mathcal{X} + d.$$

Define $T : \mathcal{X} \to \mathcal{X}$ such that

$$T(z) = \beta M z - d(z)$$

where $d(z) \in \mathcal{L}$ and $z \in \mathcal{X} + d(z)$. The expansion on $\mathcal{X}$ induced by T with digits in $\mathcal{L}$ is called the **rotational β-expansion** with parameters $(\beta, M, \mathcal{L})$.

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- If $m = 1, M = 1, \mathcal{X} = [0, 1)$ and $\mathcal{L} = \mathbb{Z}$, then the rotational beta expansion is the classical β-expansion.
- If $m = 2$, then the rotational beta expansion can be viewed as an expansion on $\mathbb{C}$ where the base is the complex number equivalent to βM. Similarly, if $m = 4$, then the rotational beta expansion can be viewed as an expansion on the set $\mathbb{H}$ of real quaternions.

Winning Result on Real Expansions

Consider the real expansion on $[0,1)$ with base $1 < b \in \mathbb{R}$ with digit set $\mathcal{D}$. For $d \in \mathcal{D}$, define

$$C_b[d] := \{x \in [0,1) : d \text{ appear in the } b-\text{expansion of } x.\}$$

Also, let $i \in \mathbb{N} \cup \{0, \infty\}$ be the length of the b-expansion of $b - \lfloor b \rfloor$, that is,

$$i := \max\{j \in \mathbb{N} : \text{ the } j\text{th digit of } b - \lfloor b \rfloor \text{ is nonzero}\}.$$

We set $K = K_b$ be the maximal length of zero blocks (blocks containing only the digit zero) among the first i digits of the b-expansion of $b - \lfloor b \rfloor$.

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Example

- $b \in \mathbb{N} \Longrightarrow i = 0 \Longrightarrow K = 0$
- $b = (1+\sqrt{5})/2 \Longrightarrow b - \lfloor b \rfloor = 010^{\infty} \Longrightarrow K = 1$

Problem: How to compute for K (e.g. $b = \sqrt{2}$)?

Let $k \in \mathbb{N}$. Denote by $\mathcal{D}_A^{k-1}$ the set of admissible blocks of length $k-1$ and suppose

$$|\mathcal{D}_A^{k-1}| = N.$$

We can write $\mathcal{D}_A^{k-1} = \{\mathbb{e}_1, \mathbb{e}_2, \ldots, \mathbb{e}_N\}$ such that

$$\mathbb{e}_1 <_{\text{lex}} \mathbb{e}_2 <_{\text{lex}} \cdots <_{\text{lex}} \mathbb{e}_N.$$

Suppose $d \in \mathcal{D}$ such that $\mathbb{e}_j d$ is admissible for all j. Then the length of cylinder set

$$\Delta(\mathbb{e}_j d) := \{z \in \mathcal{X} : \text{ the } b-\text{expansion of } z \text{ starts with } \mathbb{e}_j d\}$$

is at most b^{-k}.

Let $E_{k-1,d} := \{\mathbb{e}_j : \Delta(\mathbb{e}_j d) \text{ had length less than } b^{-k}.\}$

Lemma

Let $N \geq j \in \mathbb{N}$ such that $\mathbb{e}_j, \mathbb{e}_{j+1}, \ldots, \mathbb{e}_{j+\ell} \in E_{k-1,d}$. Then $\ell \leq K + 1$.

Theorem (Caalim-Kaneko-N., 20xx)

Suppose $K < \infty$. Let $d \in \mathcal{D}$ such that $d \le d'$ where d' is the minimal digit of $\mathbb{d}^*(1)$. Let $\alpha, \beta \in (0,1)$ such that $\log_b(\alpha\beta) \notin \mathbb{Q}$. Then $C_b[d]$ is (α, β)-winning if $\beta > A_b(\alpha)$ where

$$A_b(\alpha) := \frac{(2b(K+2)+1)\alpha - 1}{\alpha[(4b(K+2)-1) - \alpha(2b(K+2)-1)]}. \qquad (1)$$

Remark: If α is small, then $C_b[d]$ is α-winning.

Problem: What if $K = \infty$?

Idea of Proof

We have $C_b[d] = \bigcup_{k=1}^{\infty} V_k(b; d)$ where

$$V_k(b; d) = \{z \in [0, 1) : k\text{th digit of } b - \text{expansion of } z \text{ is } d.\}$$

Then

$$V_k(b; d) = \bigcup_{j=1}^{N} \Delta(\mathbb{e}_j d).$$

Assume (1) and $\log_b(\alpha\beta) \notin \mathbb{Q}$. Then there exist $n, k \in \mathbb{N}$ such that

$$2b(K+2)\alpha - \frac{4b(K+2)\alpha\beta(1-\alpha)}{1-\alpha\beta} < \frac{K+2}{\rho(\alpha\beta)^n b^{k-1}} < 1-\alpha \quad (2)$$

and so

(i) $\dfrac{K+2}{b^{k-1}} < \rho(\alpha\beta)^n(1-\alpha)$

(ii) $2\rho(\alpha\beta)^n\alpha - \dfrac{4\rho(\alpha\beta)^{n+1}(1-\alpha)}{1-\alpha\beta} < \dfrac{1}{b^k}$.

Alice must choose a_{n+1} so that $|a_{n+1} - b_n| \leq \rho(\alpha\beta)^n(1-\alpha)$. Let a be the center of an interval $\Delta(\oplus_j d)$ of $V_k(b; d)$ that is closest to b_n.

- If such interval is of length b^{-k}, i.e. $\mathbb{e}_j \notin E_{k-1,d}$, Alice chooses $a_{n+1} = a$.

- If such interval is of length less than b^{-k}, i.e. the $\mathbb{e}_j \in E_{k-1,d}$, Alice chooses $a_{n+1} = a'$ where a' is the center of an interval $\Delta(\mathbb{e}_{j'} d)$ of $V_k(b; d)$ nearest to a such that the length of the interval of $V_k(b; d)$ centered at a' is b^{-k}.

  Note that the gap between the consecutive centers of the intervals $\Delta(\mathbb{e}_j d)$ is at most $b^{-(k-1)}$.

Then, in any case, $|a_{n+1} - b_n| \le (K+2)b^{-(k-1)} < \rho(\alpha\beta)^n(1-\alpha)$ by (i) which means a_{n+1} is a valid choice.

Alice wins if she can force the situation where $A_{n+m} \subseteq V_k(b;d)$ for some $m \in \mathbb{N}$. For avoiding such a situation, Bob's best strategy is to move away as far as possible from a_{n+1} at each turn.

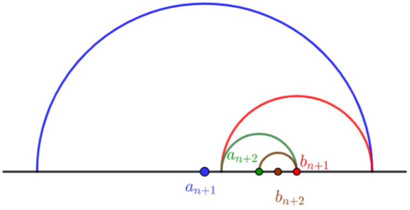


Figure: Centers $a_{n+1}, b_{n+1}, a_{n+2}, b_{n+2}$ of the balls A_{n+1}, B_{n+1}, A_{n+2} and B_{n+2}

Bob's best strategy is to choose

$$b_{n+m} = a_{n+m} + \rho\alpha(\alpha\beta)^{n+m-1}(1-\beta) \text{ for all } m \in \mathbb{N}.$$

Alice chooses $a_{n+m+1} = b_{n+m} - \rho(\alpha\beta)^{n+m}(1-\alpha)$. Then for any $m \in \mathbb{N}$ and $z \in A_{n+m+1}$, we have

$$|z - a_{n+1}| = \rho(\alpha\beta)^n\alpha - \frac{2\rho(\alpha\beta)^{n+1}(1-\alpha)[1-(\alpha\beta)^m]}{1-\alpha\beta}$$

which decreases toward $\rho(\alpha\beta)^n\alpha - \dfrac{2\rho(\alpha\beta)^{n+1}(1-\alpha)}{1-\alpha\beta}$ as m increases.

By (ii), there exists $m \in \mathbb{N}$ such that $|z - a_{n+1}| < 1/(2b^k)$ for any $z \in A_{n+m+1}$, which implies that A_{n+m+1} is contained in an interval of V_k. Therefore, $A_{n+m+1} \subseteq V_k(b; d)$.

Winning Result for Complex Expansion

Let $\xi = re^{\hat{\imath}\theta}$ where $r > 1$ and $\theta \in [0, 2\pi)$ be the base of complex expansion with fundamental domain $\mathcal{X} \cong [-1/2, 1/2)^2$ with lattice $\mathcal{L} = \mathbb{Z}[\hat{\imath}]$.

Then for $z \in \mathcal{X}$, the first digit $d(z)$ of the $E(z)$ is

$$\begin{aligned} d(z) &= \left\lfloor \mathrm{Re}(\xi z) + \tfrac{1}{2} \right\rfloor + \left\lfloor \mathrm{Im}(\xi z) + \tfrac{1}{2} \right\rfloor \hat{\imath} \\ &= \left\lfloor rx\cos(\theta) - ry\sin(\theta) + \tfrac{1}{2} \right\rfloor + \left\lfloor rx\sin(\theta) + ry\cos(\theta) + \tfrac{1}{2} \right\rfloor \hat{\imath}, \end{aligned}$$

Digit Set

We define the digit set $\mathcal{D} = \mathcal{D}(\xi) := \{d(z) : z \in \mathcal{X}\}$.

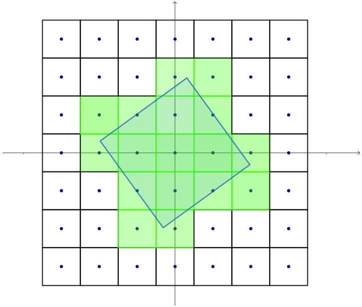


Figure: The set $\xi\overline{\mathcal{X}}$ is the blue square and $\mathcal{D}$ is the set of lattice points on the green squares.

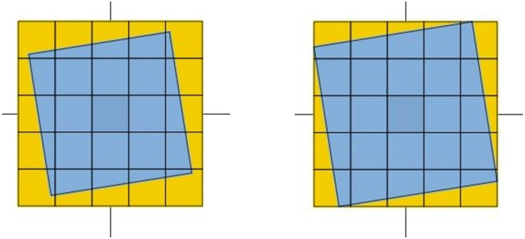
Square Digit Sets

Let $\xi = re^{\hat{\imath}\theta}$ where $\theta \in [0, \pi/4]$. Let $\mathbb{D} = \bigcup_{d \in \mathcal{D}} (\mathcal{X} + d)$.

Proposition

If $\mathbb{D}$ is a rectangle, then it is a square. Also, if $\mathbb{D}$ is a square, then there exists $N \in \mathbb{N}$ such that

$$\mathcal{D} = \{a + b\hat{\imath} : a, b \in \mathbb{Z} \text{ s.t. } |a|, |b| \leq N.\}$$



We say that ξ has a square digit set of size $N \in \mathbb{N}$ if

$$\mathcal{D} = \{a + b\hat{\imath} \mid a, b \in \mathbb{Z} \text{ such that } |a|, |b| \leq N\}.$$

If $\xi = re^{\hat{\imath}\theta}$ has a square digit set of size N, then

$$N = \left\lceil \frac{r(c+s)+1}{2} \right\rceil - 1 \text{ where } c = \cos\theta \text{ and } s = \sin\theta.$$

Proposition

The base ξ has a square digit set of size $N \in \mathbb{N}$ if and only if

$$(2N-1)(c+s) < r \leq \frac{2N+1}{c+s} =: u_N(\theta).$$

Theorem

Let $\theta \in [0, \pi/4]$ *and* $K(\theta) := \lceil (sc+1)/(2sc) \rceil - 1$*. Then* $\xi = re^{\hat{\imath}\theta}$ *has a square digit set if and only if*

$$r \in \bigcup_{N=1}^{K(\theta)} \left((2N-1)(c+s), u_N(\theta)\right].$$

Note that by symmetry, we can restrict $\theta \in [0, {}^{\pi}/_{2}]$ (except on some rare cases). Further, we can consider $\theta \in [0, {}^{\pi}/_{4}]$ such that if $\theta \in ({}^{\pi}/_{4}, {}^{\pi}/_{2}]$, then sin and cos switch roles.

Property (C_k)

Let $k \in \mathbb{N}$. We say that the ξ-expansion has property (C_k) if

$$\xi^{-k}\mathcal{X} + \xi^{-(k-1)}a_{k-1} + \cdots + \xi^{-1}a_1 \subseteq \mathcal{X}$$

for any admissible block $a_1a_2\cdots a_{k-1} \in \mathcal{D}^{k-1}$ with
$(C_1) : \xi^{-1}\mathcal{X} \subseteq \mathcal{X}$.

Problem: When does (C_k) hold?

Observe that

$$(C_k) \text{ holds } \iff \xi^{-k}\mathcal{X} + \sum_{j=1}^{k-1} \xi^{-j} a_j \subseteq \mathcal{X}$$

for all admissible $a_1 a_2 \cdots$.

$$\iff t_P := \xi^{-k} P + \sum_{j=1}^{k-1} \xi^{-j} a_j \in \overline{\mathcal{X}}$$

where P is a corner point of $\overline{\mathcal{X}}$

If $\xi = re^{\hat{\imath}\theta}$ has square digit set of size N, then

$$-A(N,k) \leq \mathrm{Re}(t_P), \mathrm{Im}(t_P) \leq A(N,k)$$

where $A(N,k) = \dfrac{|c^{(k)}| + |s^{(k)}|}{2r^k} + N \displaystyle\sum_{j=1}^{k-1} \dfrac{|c^{(j)}| + |s^{(j)}|}{r^j}$ such that $c^{(j)} = \cos(j\theta)$. So $A(N,k) < 1/2$ implies (C_k).

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Let $f_N^{(k)}(r) := r^k - 2N \displaystyle\sum_{j=1}^{k-1} r^{k-j}(|c^{(j)}| + |s^{(j)}|) - (|c^{(k)}| + |s^{(k)}|)$

such that $f_N^{(k)}(r) > 0 \iff A(N,k) < 1/2$.
Then $f_N^{(k)}$ has a unique positive root $v_N^{(k)} := v_N^{(k)}(\theta)$. Note that $v_N^{(k)}$ increases as k increases.

Theorem

Let $\theta \in [0, \pi/4]$ and $r > 1$. Let $N \in \mathbb{N}$. Then $\xi = re^{\hat{\imath}\theta}$ has a square digit set of size N and (C_n) holds for the ξ-expansion for $n \in \{1, 2, \ldots, k\}$ if

$$v_N^{(k)}(\theta) < r < u_N(\theta).$$

In particular, (C_2) holds for the ξ-expansion if and only if $v_N^{(2)}(\theta) < r < u_N(\theta)$.

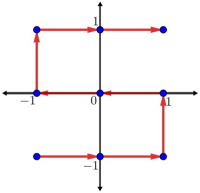
Let $C_\xi[0] := \{z \in \mathcal{X} :$ the ξ-expansion of z has digit $0\}$. Observe that $C_\xi[0] = \bigcup_{k=1}^{\infty} V_k(\xi, 0)$ where $V_k := \{z \in \mathcal{X} : d_k(z) = 0\}$.

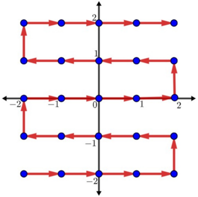
Proposition

Let $k \in \mathbb{N}$ and $\mathbb{a} = a_1 a_2 \cdots a_k \in \mathcal{D}^k$ be an admissible block. If (C_n) holds for $n \in \{1, 2, \ldots, k+1\}$, then $(\mathbb{a}, 0)$ is admissible. Moreover, $V_{k+1}(\xi; 0)$ can be partitioned into disjoint squares as

$$V_{k+1}(\xi; 0) = \bigcup_{a_1 a_2 \cdots a_k \in \mathcal{D}^k \text{ is admissible}} \xi^{-(k+1)}\mathcal{X} + \sum_{j=1}^{k} \xi^{-j} a_j.$$

Also, if ξ has square digit set, the we can define an ordering on $\mathcal{D} = \{w_1, w_2, \ldots, w_{(2N+1)^2}\}$ such that $|w_{\ell+1} - w_\ell| = 1$ for all ℓ.





This translates to an order of the centers of the form $\displaystyle\sum_{j=1}^{k-1} \xi^{-j} a_j$ of the squares that make up V_k. Hence, the distance between "consecutive" centers of V_k is at most $\dfrac{\sqrt{2}}{r^{k-1}}$.

Theorem (Caalim-Kaneko-N., 20xx)

Let $\xi = re^{\hat{\imath}\theta}$. Let $N, k \in \mathbb{N}$ and suppose $v_N^{(k)}(\theta) < r < u_N(\theta)$. Let $0 < \alpha, \beta < 1$ and $\rho > 0$. Suppose

$$\beta > F_r(\alpha) := \frac{(2\sqrt{2}r + 1)\alpha - 1}{\alpha[(1 - 2\sqrt{2}r)\alpha + (4\sqrt{2}r - 1)]}.$$

and ρ satisfies

$$2\alpha - \frac{4\alpha\beta(1-\alpha)}{1-\alpha\beta} \leq \frac{1}{\rho(\alpha\beta)^n r^k} < \frac{1-\alpha}{\sqrt{2}r}$$

for some $n \in \mathbb{N}$, then $C_\xi[0]$ is (α, β, ρ)-winning.

Losing Result

Consider the rotational expansion on $\mathcal{X} \subseteq \mathbb{R}^m$ with parameters $(b, M, \mathcal{L})$.

Theorem

Assume that $\mathcal{X}$ has a nonzero interior point. Let $\Omega = d_1 d_2 \cdots d_n \in \mathcal{L}^n$ be an admissible word of length n There exists a positive constant C_Ω such that for any real number $\alpha \in (0, 1)$ so that $C_\Omega |b|^{-n} \le \alpha$ and $|b|^{-n} < \alpha$, we have

$$\mathcal{C}[\Omega] := \{z \in \mathcal{X} : \Omega \text{ appears in the expansion of } z\}$$

is (α, β)-losing, where $\beta = \alpha^{-1}|b|^{-n} < 1$.

Problem: Minimize C_Ω.

-END-

-Thank you!-

Alice to Bob when $S = C_b[0]$ and α is small:

References

[1] Wolfgang M. Schmidt.
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