Linear independence of certain infinite series through the application of word combinatorics

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Motivation

$\xi = \sum_{n=-N(\xi)}^{\infty} t_n b^{-n}$: base-b expansion of $\xi \in \mathbb{R}_{>0}$. We define

$$\lambda(b,\xi,N) := \#\{n \in \mathbb{N} \mid n \le N,\ t_n \neq 0\} \quad \text{i.e.} \quad \#\text{ of nonzero digits}.$$

By the theorem of Adamczewski and Faverjon(2012), Bugeaud(2012), we have

$$\liminf_{N\to\infty} \frac{\lambda(b,\xi,N)}{N^{1/\ell}} = 0 \implies \xi \notin \overline{\mathbb{Q}} \text{ or } \deg \xi \ge \ell + 1.$$

The result above is a rediscovery by Erdős-Straus (1954), Erdős (1957).

Question (Border case)

We consider $\xi \in \mathbb{R}_{>0}$ s.t. $\lambda(b,\xi,N) \asymp N^{1/\ell}$.

Moreover, we give sufficient conditions for the linear independence of

$$\{\xi_1^{d_1}\xi_2^{d_2}\xi_2^{d_3} \mid (d_1,d_2,d_3) \in B\} \text{ for } B \subset \mathbb{Z}_{\ge 0}^3.$$

Transcendental numbers x_0, x_1, x_2 $(\ell = 3)$

By the Thue-Siegel-Roth theorem, the following numbers x_0, x_1, x_2 are transcendental (T.Tsurumaki, 2025+).

Definition

Let S_d $(d = 0, 1, 2)$ *be* **the set of all non-negative binary integers where the digits at positions not congruent to d (mod 3) are 0**.
Example,

$$\begin{aligned} S_0 &= \{(0)_2, (1)_2, (1000)_2, (1001)_2, (1000000)_2, (1000001)_2, (1001000)_2 \cdots\} \\ &= \{0, 1, 8, 9, 64, 65, 72, \cdots\} \end{aligned}$$

$$x_d := \sum_{n \in S_d} \frac{1}{b^n} \ (d = 0, 1, 2), \qquad \textit{Note that } \lambda(b, x_d, N) \asymp N^{1/3}. \ (\textit{thus, } \ell = 3).$$

- Modifying the Erdős-Straus method, lin. indep. is obtained for x_0, x_1, x_2 in the following cases: $B = \{(d_1, d_2, d_3) \mid d_1, d_2, d_3 \geq 0, d_1 + d_2 + d_3 \leq 2\}$.
- However, $x_0 x_1 x_2 = 2$.

Main result

Let $P(X_1, X_2, X_3) = a_0 + a_1 X_1^3 + a_2 X_2^3 + a_3 X^3$ and $\lambda(b, \xi_i, N) \asymp N^{1/3}$.

$$\text{Goal}: \ P(\xi_0, \xi_1, \xi_2) \neq 0.$$

$\deg P = 3 \implies \xi_1, \xi_2, \xi_3$ are border cases. Using word combinatorics, we have:

Theorem (O. ,2025+, generalization of x_0, x_1, x_2)

$(t_d(n))_{n \geq 0} \ \in \mathbb{Z}_{>0}$ $(d \in \{0, 1, 2\})$ *satisfying the following:*

$$t_d(n) \leq e^{C_0(b)n} \quad (d \in \{0, 1, 2\}, \ n \gg 1),$$

where $C_0(b) := (\log b)/29$. *We define the numbers* $\widetilde{x_0}, \widetilde{x_1}, \widetilde{x_2}$:

$$\widetilde{x_d} := \sum_{n \in S_d} \frac{t_d(n)}{b^n} \quad (d \in \{0, 1, 2\}).$$

Then, the set $\{1, \widetilde{x_0}^3, \widetilde{x_1}^3, \widetilde{x_2}^3\}$ *is linearly independent over* $\mathbb{Q}$.

Our method does not need a functional equation (flexible).