Digit Expansions in Complex Bases

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Abstract

We consider a numeration system where the base is a complex Gaussian rational, and define expansions for complex numbers. They exhibit interesting shapes and are related to fractal tilings.

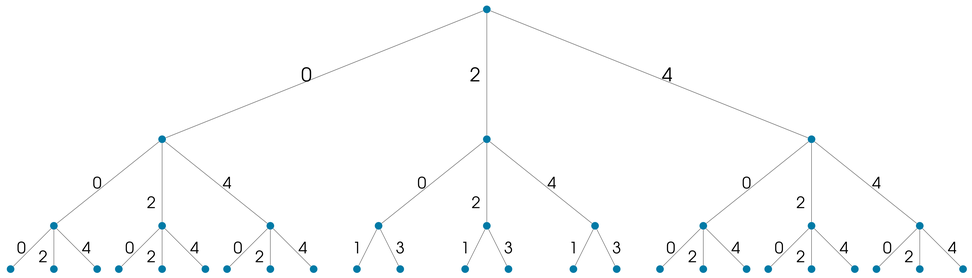
Background

- **Base**: $\alpha \in \mathbb{Q}(i)$, $|\alpha| > 1$. Minimal polynomial $a_2x^2 + a_1x + a_0$.
- **Digit set**: $\mathcal{D} = \{0, 1, \dots, |a_0| - 1\}$.
- **Lattice**: $\Lambda_\alpha = a_2\mathbb{Z} + (a_2\alpha + a_1)\mathbb{Z}$ (will play the role of the "integers").
  The key is that $\mathcal{D}$ is a residue set for $\Lambda_\alpha/\alpha\Lambda_\alpha$.

Algorithm

Let $z \in \Lambda_\alpha$. Find the unique $d_0 \in \mathcal{D}$ and $z_1 \in \Lambda_\alpha$ such that $z = \alpha z_1 + d_0$. Then, d_0 will be the digit at the units place.
For $k \geq 2$, set $z_k \in \Lambda_\alpha$ by $z_{k-1} = \alpha z_k + d_{k-1}$. If we eventually reach $z_k = 0$ (we assume that α has this property for every starting point z) then z has an **integer** α**-expansion**

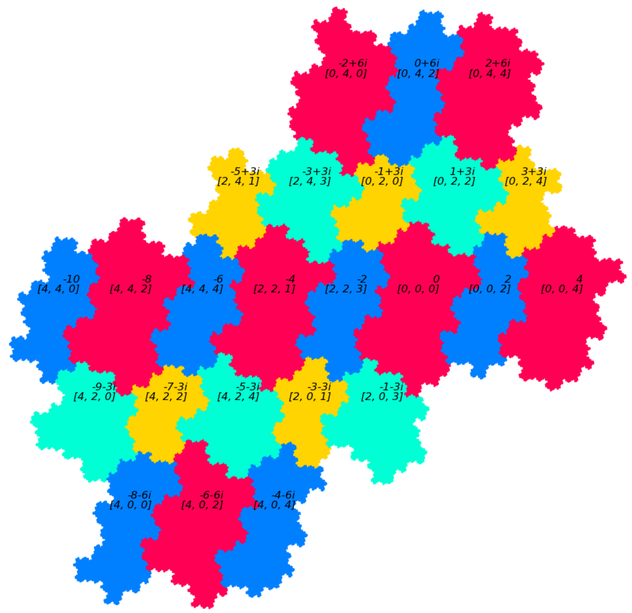
$$z = \alpha^k d_k + \cdots + \alpha^2 d_2 + \alpha d_1 + d_0 =: (d_k \dots d_0)_\alpha.$$



Integer α-expansions can be traced as finite paths on a tree starting from the root. The graph above illustrate the example $\alpha = \frac{-1+3i}{2}$, root of $2x^2 + 2x + 5$, and digits $\mathcal{D} = \{0, 1, 2, 3, 4\}$. Not all digit combinations are allowed; in this case, the two left-most digits are even. For example, $-3 + 3i = 2\alpha^2 + 4\alpha + 3 = (243)_\alpha$.

Expansions of complex numbers

An α**-expansion** of $x \in \mathbb{C}$ is an expansion of the form $x = \sum_{j \leq k} d_j \alpha^j$ with $d_j \in \mathcal{D}$ such that $(d_k \dots d_l)_\alpha$ corresponds to the integer α-expansion of some $z \in \Lambda_\alpha$ for every $l \leq k$. We denote it $x = (d_k \dots d_0.d_{-1}d_{-2}\dots)_\alpha$. They correspond to infinite paths on the tree.



Let $\alpha = \frac{-1+3i}{2}$; the corresponding lattice is $\Lambda_\alpha = 2\mathbb{Z} + (1 + 3i)\mathbb{Z}$. We illustrate the 24 points $z \in \Lambda_\alpha$ whose integer α-expansion $[d_2, d_1, d_0]$ is found on the tree. Around each point there is a fractal tile, which consists of all $x \in \mathbb{C}$ of the form $x = (d_2 d_1 d_0.\cdots)_\alpha$.

Main Results

- The α-expansion of a complex number is unique almost everywhere.
- An expansion $x = \sum_{j \leq k} d_j \alpha^j$ is an α-expansion of $x \in \mathbb{C}$ if and only if the series conveges to 0 in each p-adic completion of $\mathbb{Q}(i)$ whenever p is a Gaussian prime dividing the denominator of α.

Conjecture

Given a base α, almost every complex number is normal in this numeration system.

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