Frostman and Fourier characterisation of fractal dimensions

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Known results — Hausdorff dimension

- Let $\mathscr{P}(E)$ denote the collection of Borel probability measures supported on E.
- (Frostman's Lemma) Let E be a Borel set in $\mathbb{R}^d$. If $0 \le s < \dim_H E$, then there exist $\mu \in \mathscr{P}(E)$ and a constant $c > 0$ such that $\mu(B(x,r)) \le cr^s$ for $x \in \mathbb{R}^d$ and $0 < r < 1$.
- (Mass distribution principle) Let μ be a Borel measure supported on E. Suppose for some $s > 0$, there is a constant $c > 0$ such that $\mu(B(x,r)) \le cr^s$ for $x \in \mathbb{R}^d$ and $0 < r < 1$. Then $\dim_H E \ge s$.
- Equivalent definition of Hausdorff dimension:

$$\begin{aligned}\dim_H E =& \sup\{s : \text{there exist } \mu \in \mathscr{P}(E) \text{ and } c > 0 \text{ such that } \mu(B(x,r)) \le cr^s \\ &\qquad \text{for all } x \in \mathbb{R}^d \text{ and } 0 < r < 1\} \\ =& \sup_{\mu \in \mathscr{P}(E)} \liminf_{r \to 0} \inf_{x \in E} \frac{\log \mu(B(x,r))}{\log r}.\end{aligned}$$

Known results — Packing dimension

- (Cutler,1995) Let E be a non-empty compact subset of $\mathbb{R}^d$. For every $0 \leq s < \dim_P E$, there exist $\mu \in \mathscr{P}(E)$ and $c > 0$ such that, for each $x \in \mathbb{R}^d$, there is $\{r_n(x)\}_n \downarrow 0$ such that $\mu(B(x, r_n(x))) \leq c r_n(x)^s$.
- If there exists a Borel measure μ with $\mu(E) > 0$ such that for all $x \in E$, $\limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s$, then $\dim_P(E) \geq s$.
- Equivalent definition of Packing dimension:

$$\begin{aligned}\dim_P(E) = \sup\{s \geq 0 : & \text{ there exist } \mu \in \mathscr{P}(E), c > 0 \text{ such that for each} \\ & x \in \mathbb{R}^d, \text{there is } \{r_n(x)\}_n \downarrow 0 \text{ such that } \mu(B(x, r_n(x))) \leq c r_n(x)^s \\ = \sup_{\mu \in \mathscr{P}(E)} \inf_{x \in E} & \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.\end{aligned}$$

Main result

Theorem (Falconer-Z.,2025+)

Let $E \subset \mathbb{R}^d$ be compact. Then

$$\begin{aligned}\underline{\dim}_{MB} E = \sup\Big\{ s \geq 0 : &\text{ for any sequence } \{r_k\}_k \downarrow 0 \text{ there exist } \{r_{k_i}\}_i \subseteq \{r_k\}_k \\ &\text{and a measure } \mu \in \mathcal{P}(E) \text{ such that } \mu(B(x, r_{k_i})) \leq r_{k_i}^s \\ &\text{for all } x \in \mathbb{R}^d \text{ and } i \in \mathbb{N} \Big\} \\ = \inf_{\{r_k\}\searrow 0} \sup_{\mu\in\mathcal{P}(E)} &\limsup_{k\to\infty} \inf_{x\in E} \frac{\log \mu(B(x, r_k))}{\log r_k}.\end{aligned}$$

An independent sequence

Theorem (Falconer-Z.,2025+)

Let E be a non-empty compact set in $\mathbb{R}^d$. Then

$$\overline{\dim}_C(E) = \sup\{s : \textit{there exist } \mu \in \mathcal{P}(E), c > 0 \textit{ and } \{r_n\}_n \downarrow 0 \textit{ such that for all } x \in \mathbb{R}^d, \mu(B(x, r_n)) \leq c r_n^s\}$$

$$= \sup_{\mu \in \mathcal{P}(E)} \limsup_{r \to 0} \inf_{x \in E} \frac{\log \mu(B(x,r))}{\log r}.$$

- $\underline{\dim}_{MB} E \leq \overline{\dim}_C(E) \leq \dim_P E$.

Example (Falconer-Z.,2025+)

Given $0 < a < b < c < 1$, there exists a compact set E such that

$$\underline{\dim}_{MB} E = a, \quad \overline{\dim}_C(E) = b, \quad \dim_P E = c.$$

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Proposition

Let $0 < \epsilon < 1$ and $\rho > 0$. There are constants $b_1, b_2, r_0 > 0$ depending on d, ϵ and ρ such that for all probability measures μ on $\mathbb{R}^d$ with support in $B(0, \rho)$ and all $r \leq r_0$,

$$b_1 r^{d(1+\epsilon)} \int_{|z| \leq r^{-1}} |\widehat{\mu}(z)|^2 dz \leq \int \mu(B(x, r)) d\mu(x) \leq b_2 r^{d(1-\epsilon)} \int_{|z| \leq r^{-1}} |\widehat{\mu}(z)|^2 dz.$$

Proposition

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Correlation dimensions:

$$\overline{\dim}_C E = \sup_{\mu\in\mathcal{P}(E)} \liminf_{R\to\infty} \frac{\log\left(R^{-d}\int_{|z|\leq R} |\widehat{\mu}(z)|^2 dz\right)}{-\log R}$$

$$\dim_H E = \underline{\dim}_C E = \sup_{\mu\in\mathcal{P}(E)} \limsup_{R\to\infty} \frac{\log\left(R^{-d}\int_{|z|\leq R} |\widehat{\mu}(z)|^2 dz\right)}{-\log R}.$$

Integral form

- Box dimensions:

$$\underline{\dim}_B E = \liminf_{r \to 0} \sup_{\mu \in \mathcal{P}(E)} \frac{\log \int \mu(B(x,r)) d\mu(x)}{\log r}$$

$$\overline{\dim}_B E = \limsup_{r \to 0} \sup_{\mu \in \mathcal{P}(E)} \frac{\log \int \mu(B(x,r)) d\mu(x)}{\log r}.$$

- Modified lower box dimension:

$$\underline{\dim}_{MB} E = \inf_{\{r_k\} \searrow 0} \sup_{\mu \in \mathcal{P}(E)} \limsup_{k \to \infty} \frac{\log \int \mu(B(x,r_k)) \mathrm{d}\mu(x)}{\log r_k}.$$

Fourier characterisations of dimensions

- Box dimensions:

$$\underline{\dim}_B E = \liminf_{R\to\infty} \sup_{\mu\in\mathcal{P}(E)} \frac{\log\left(R^{-d}\int_{|z|\le R}|\widehat{\mu}(z)|^2 dz\right)}{-\log R},$$

$$\overline{\dim}_B E = \limsup_{R\to\infty} \sup_{\mu\in\mathcal{P}(E)} \frac{\log\left(R^{-d}\int_{|z|\le R}|\widehat{\mu}(z)|^2 dz\right)}{-\log R}.$$

- Modified lower box dimension:

$$\underline{\dim}_{MB} E = \inf_{\{R_k\}\nearrow\infty} \sup_{\mu\in\mathcal{P}(E)} \limsup_{k\to\infty} \frac{\log\left(R_k^{-d}\int_{|z|\le R_k}|\widehat{\mu}(z)|^2 dz\right)}{-\log R_k}.$$

More details on my poster.

Thank you for your attention!