The distribution of minimal fractions on the Stern-Brocot tree

Lama Tarsissi
Joint work: Sudarshane Shinde, and Mohammad Nasr

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Outline

1. Combinatorics on Words and Terminology
2. First and second orders of balancedness
3. Characterisation of Minimal paths
4. Distribution of minimal fractions at a certain level of $SB_{/3}$ tree
5. Conclusion and Perspectives

Terminology

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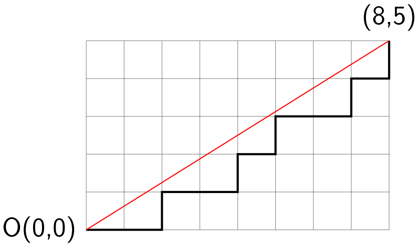

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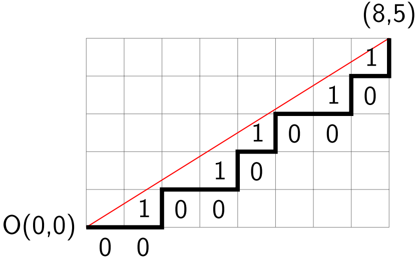

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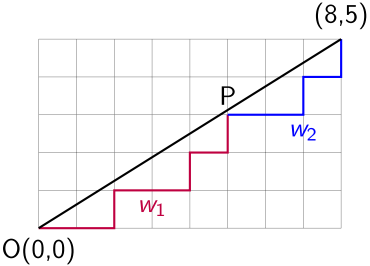

- P the closest point to the line segment on the Christoffel path.

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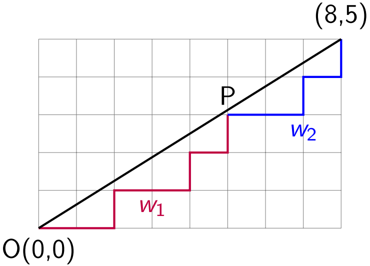

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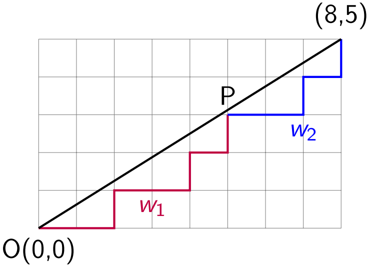

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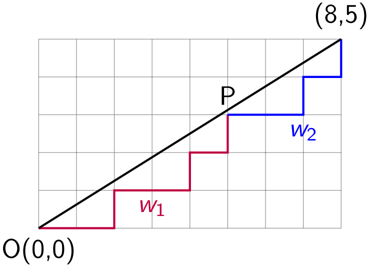

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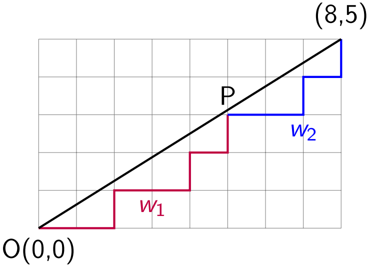

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The uniqueness of this points is due to Borel, Laubie (BL1993).

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Example

Let the Christoffel word of slope $3/5$;

$$\begin{tikzcd}[column sep=tiny] & \mathrm{C}(3/5)=(001,00101) \arrow[dl, "L"'] \arrow[dr, "R"] & \\ \mathrm{C}(4/7)=(001,00100101) & & \mathrm{C}(5/8)=(00100101,00101) \end{tikzcd}$$

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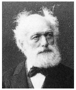

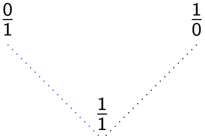

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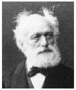

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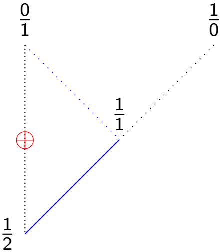

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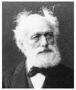

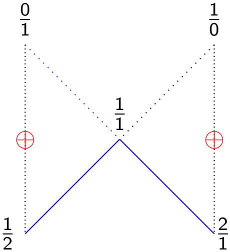

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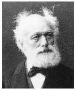

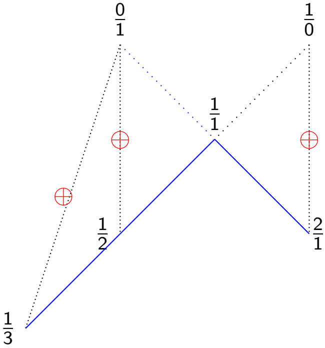

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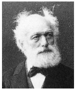

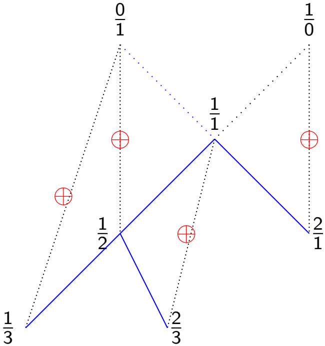

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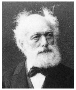

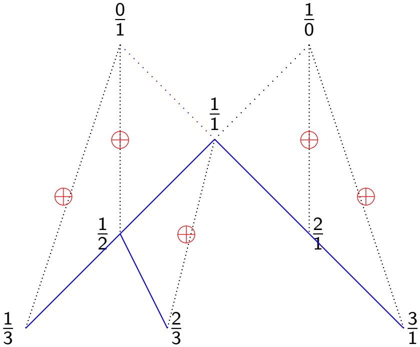

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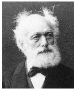

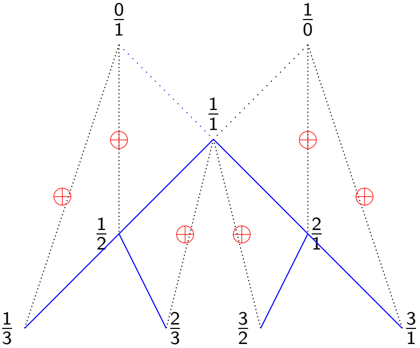

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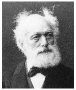


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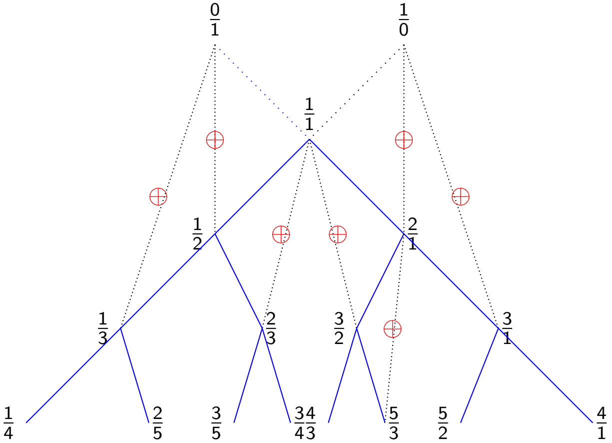

Continued fractions

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The continued fraction of a rational number $\frac{a}{b}$ is given by: $\frac{a}{b} = [a_0, \ldots, a_n]$, s.t:

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Example

We write $\frac{5}{8} = [0, 1, 1, 1, 2]$ since:

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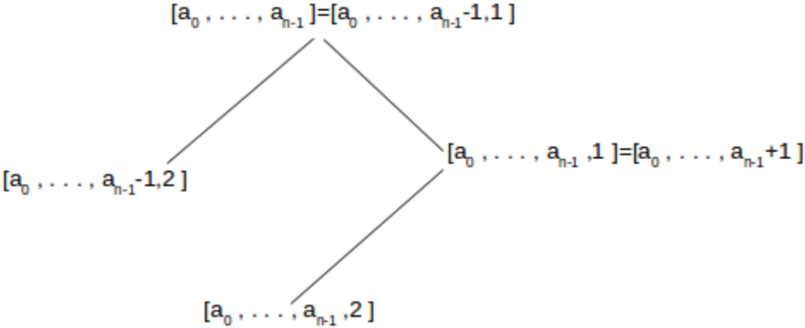

Some Relations

Lemma (Smith 1876)

Let $\frac{a}{b} = [a_0, a_1, \ldots, a_z]$, *the* *directive sequence* *of* a/b *is given by:*
$\Delta(a/b) = 1^{a_0}0^{a_1}\ldots x^{a_{z-1}}y^{a_z-1}$, *where* $x \neq y$ *and* $x, y \in A$.

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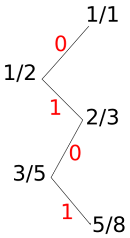
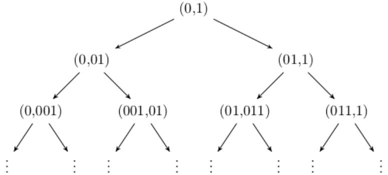
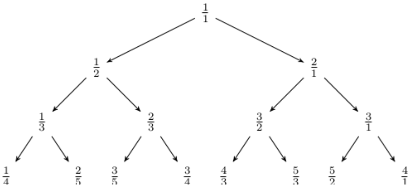


Isomorphism between the two trees



First 3 levels of the Christoffel tree



First 4 levels of the Stern-Brocot tree

Balanced words

Definition

A word w is $\delta-$balanced if for all $a \in A$, and for all two factors w' and w'' of w, such that $|w'| = |w''|$, we have:

$$||w'|_a - |w''|_a| \leq \delta.$$

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Theorem (Berstel, de Luca BdL1997)

A word w is a Christoffel word iff it is a $1-$balanced word of the form $0v1$, where v is a palindrome.

Balanced matrix

For all $1 \leq i, j \leq n$, we have: $S_w[i,j] = |w[j \ldots i+j-1]|_1$.

Definition (Balanced Matrix, Tarsissi 2017)

$B_w[i,j] = S_w[i,j] - \min(S_w[i])$, $\forall$ $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

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Second order balance property on Christoffel words

Lama Tarsissi[1] * and Laurent Vuillon[2]

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The second order balance matrix is given by:

$$U_w = (u_{ij})_{1\leq i,j\leq n-1}; \quad U_w[i,j] = \max(B_{B_w[i]}[j]).$$

1. The second order of balancedness is: $\delta^2(w) = \max(U_w)$.
2. The matrix $U_{\frac{a}{b}}$ has 3 axis of symmetry: Horizontal, vertical and diagonal.
3. $U_{\frac{a}{b}} = U_{\frac{b}{a}}$.

Recursive construction

$$\textit{For } \frac{a}{b} = [a_0, \cdots, a_z] \Longrightarrow U_{\frac{a}{b}} = \left(\begin{array}{c|c|c} \alpha & \cdot & \cdot \\ \hline \gamma & \beta & \cdot \\ \hline \cdot & \cdot & \cdot \end{array}\right),$$

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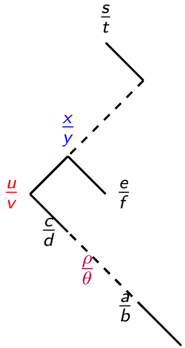

- $\frac{u}{v} = [a_0, \cdots, a_{z-1} + 1]$;
- $\frac{c}{d} = [a_0, \cdots, a_{z-1}, 2]$;
- $\frac{x}{y} = [a_0, \cdots, a_{z-1}]$;
- $\frac{e}{f} = [a_0, \cdots, a_{z-1} - 1, 2]$;
- $\frac{s}{t} = [a_0, \cdots, a_{z-2}]$;
- $\frac{\rho}{\theta} = [a_0, \cdots, a_z - 2]$

Example

$$U_{\frac{5}{8}} = \left(\begin{array}{ccccc|cc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
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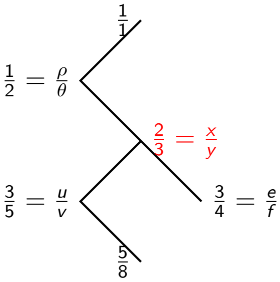
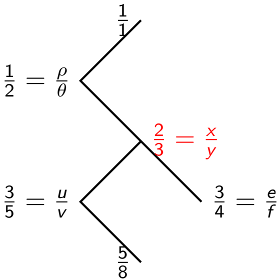
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right).$$

Example



$\frac{1}{1}$
$\frac{1}{2} = \frac{\rho}{\theta}$
$\frac{2}{3} = \frac{x}{y}$
$\frac{3}{5} = \frac{u}{v}$
$\frac{3}{4} = \frac{e}{f}$
$\frac{5}{8}$

Example



Top left part of $U_{\frac{3}{5}}$

$\gamma(U_{\frac{3}{4}})$

$U_{\frac{1}{2}} + 1$

$$U_{\frac{5}{8}} = \left(\begin{array}{ccccc|cc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
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1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right).$$

$$U(\tfrac{1}{2}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};$$

The $\beta-$block

$$U(\frac{3}{4}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \color{red}1 & \color{red}1 & \color{red}2 & \color{red}2 & \color{red}1 & 1 \\ \color{red}1 & \color{red}1 & \color{red}2 & \color{red}2 & \color{red}1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The $\gamma -$ block

$$U(\tfrac{3}{5}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \color{red}1 & \color{red}1 \\ 1 & 1 & 1 & 2 & 1 & \color{red}1 & \color{red}1 \\ 1 & 1 & 1 & 1 & 1 & \color{red}1 & \color{red}1 \\ 1 & 2 & 1 & 2 & 1 & \color{red}2 & \color{red}1 \\ 1 & 1 & 1 & 1 & 1 & \color{red}1 & \color{red}1 \\ \color{red}1 & \color{red}1 & \color{red}1 & \color{red}2 & \color{red}1 & \color{red}1 & \color{red}1 \\ \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 \end{pmatrix}$$

The $\alpha-$block

$$\delta^2(\frac{5}{8}) = 2.$$

Some results on δ^2

Theorem (Tarsissi,F 20)

Let $\frac{a}{b} = [a_0, a_1, \cdots, a_z]$*. Then we have the following.*

$$\delta^2\left(C\left(\frac{a}{b}\right)\right) = \begin{cases} \delta^2\left(C\left(\frac{u}{v}\right)\right) & \text{if } a_{z-1} \geq 2 \text{ and } a_z = 2, \\ \delta^2\left(C\left(\frac{\rho}{\theta}\right)\right) + 1 & \text{elsewhere.} \end{cases}$$

where: $\frac{u}{v} = [a_0, a_1, \cdots, a_{z-1} + 1]$; *and* $\frac{\rho}{\theta} = \begin{cases} [a_0, a_1, \cdots, a_z - 2] & \text{if } a_z \geq 4 \\ [a_0, a_1, \cdots, a_{z-1} + 1] & \text{if } a_z = 3 \\ [a_0, a_1, \cdots, a_{z-2}] & \text{if } a_z = 2. \end{cases}$

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Theorem (Tarsissi,F 20)

For each level k in the SB tree, we have: $\delta_k^2 \geq \lceil \frac{k}{3} \rceil$.

Minimal paths

We assign for each fraction in SB_L tree the δ^2 value of its related Christoffel word

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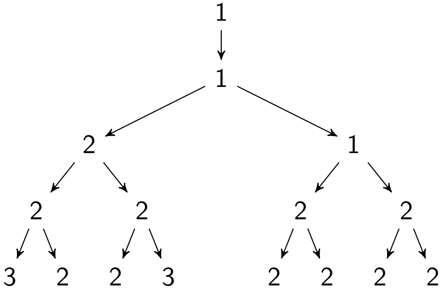

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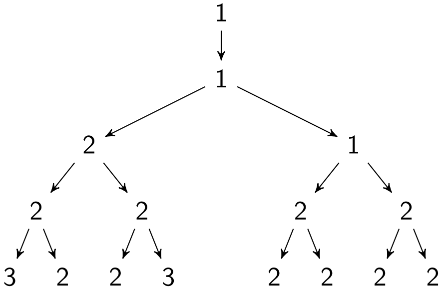

1. $P(\frac{a}{b})$ the sequence of all the fractions in the path from the root of SB_L to the element $\frac{a}{b}$,
2. $P_{\delta^2}(\frac{a}{b})$ the sequence of the related δ^2 values.

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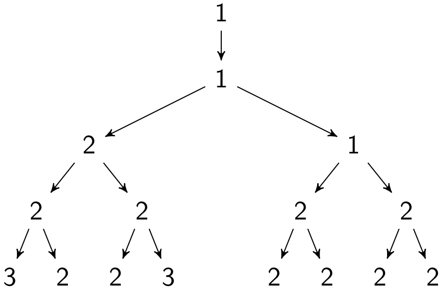

Example

$P(\frac{7}{12}) = (\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12})$, and the related δ^2 sequence is
$P_{\delta^2}(\frac{7}{12}) = (1, 1, 1, 2, 2, 2)$.

Minimal paths

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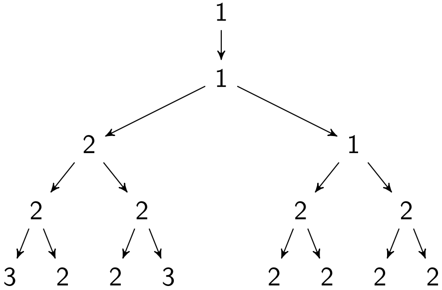

Definition (Tarsissi,F 20)

For each level k of the SB_L, the minimal path MP_k is given by:
$(1, 1, 1, 2, 2, 2, 3, \dots, \lceil \frac{k}{3} \rceil)$.
A minimal fraction $\frac{a}{b}$ has $P_{\delta^2}(\frac{a}{b}) = (1, 1, 1, 2, 2, 2, 3, \dots, \lceil \frac{k}{3} \rceil)$.

A minimal path

Definition

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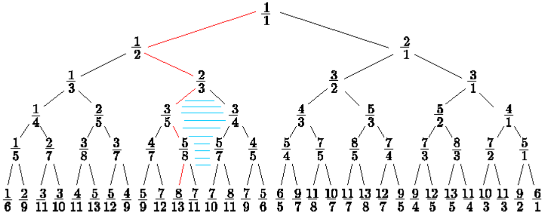

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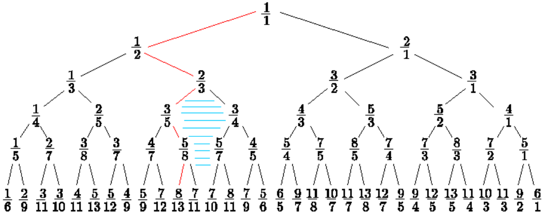

Lemma

The zigzag path who converges to the inverse of the golden ratio, $\frac{1}{\varphi} = \varphi - 1$, minimizes the growth of δ^2 on the Stern-Brocot tree.

Characterisation of some minimal paths

Theorem (Tarsissi,F 20)

Let $\frac{a}{b}$ be a minimal fraction at level $k = 3n$ with $n \geq 1$. Then the continued fraction representation of $\frac{a}{b}$ is of the form $[0, 1, P(\Omega), 2]$ where $P(\Omega)$ is a concatenation of elements of $\Omega = \{\omega_0 = (1, 2), \omega_1 = (1, 1, 1), \omega_2 = (2, 1), \omega_3 = (3)\}$ *and conversely.*

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The continued fraction of the minimal fraction $\frac{8}{13}$ is $[0, 1, 1, 1, 1, 2]$ and satisfies the conditions.

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The continued fraction of the minimal fraction $\frac{8}{13}$ is $[0, 1, 1, 1, 1, 1, 2]$ and satisfies the conditions.

Corollary

Let k be a certain level in SB_L, the number of MP_k is equal to:

$$\begin{cases} 4^{\frac{k}{3}-1} & \text{if } k \equiv_3 0 \\ 2 \cdot 4^{\lfloor \frac{k}{3} \rfloor -1} & \text{if } k \equiv_3 1 \\ 4 \cdot 4^{\lfloor \frac{k}{3} \rfloor -1} & \text{if } k \equiv_3 2 \end{cases}$$

Question:

How are the minimal fractions and minimal paths distributed on the SB tree?

Construction of $SB_{/3}$ tree from SB tree

1. At level one, we put the root $\frac{2}{3}$.
2. For level two, we choose 8 descendants of $\frac{2}{3}$ coming from level 6 of SB_L.
3. For level k, we choose 8^{k-1} descendants coming from level $3k$ of SB_L tree.

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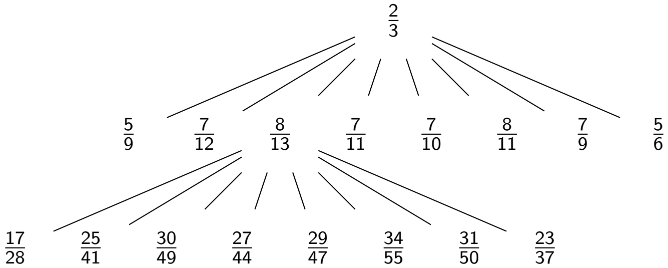
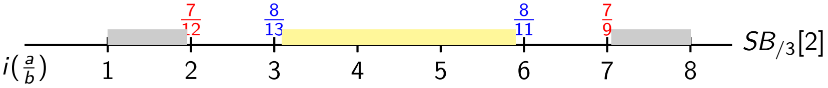


Figure: Partial $SB_{/3}$ up to level 3

Minimal fractions at level 2

1. $SB_{/3}[k]$ be the ordered set of fractions in $SB_{/3}$ at level k
2. $i(\frac{a}{b})$ denote the position (index) of the fraction $\frac{a}{b}$ in $SB_{/3}[k]$.



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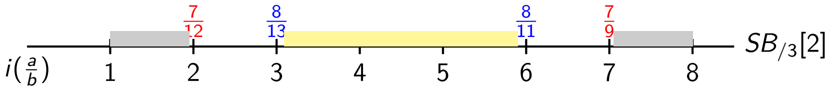

3. Let f_k^ℓ and f_k^r denote the leftmost and the rightmost minimal fraction respectively appearing at level k with respective positions i_k^ℓ and i_k^r.
4. Let $f_k^{c\ell}$ and f_k^{cr} denote the center left and the center right minimal fraction respectively appearing aroud the biggest gap.

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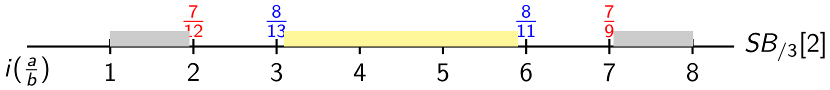

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4. Let $f_k^{c\ell}$ and f_k^{cr} denote the center left and the center right minimal fraction respectively appearing aroud the biggest gap.
5. Let g_k^ℓ and g_k^r be the the leftmost and rightmost resp. gaps n level k.
6. Let g_k denote the maximum gap between two consecutive minimal fractions at level k; $g_k = i_k^{cr} - i_k^{c\ell} - 1$.

Minimal fractions position

Proposition (Tarsissi,S,N 2025+)

Let f be a minimal fraction in $SB_{/3}[k]$ and let $c_1, \ldots, c_8$ be its ordered children at level $k+1$. Then c_2, c_3, c_6 and c_7 are minimal.

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$[0, 1, P(\Omega), \omega_0, 2], [0, 1, P(\Omega), \omega_1, 2], [0, 1, P(\Omega), \omega_2, 2]$, *or* $[0, 1, P(\Omega), \omega_3, 2]$.

Minimal fractions position

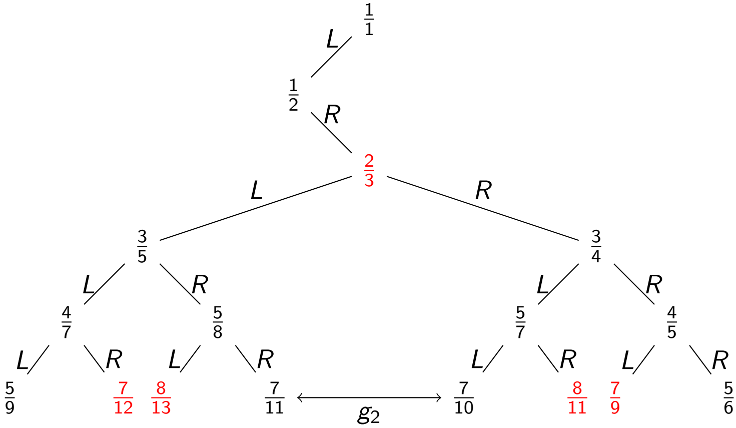

Figure: Partial SB_L containing $\frac{2}{3}$ up to level 6 along with minimal fractions highlighted in red with the left and right sense of descent. The arrow g_2 represents the gap $g_2 = 2$.

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Theorem (Tarsissi,S,N, 2025+)

Let $g_1^\ell = g_1^r = g_1 = 0$. Then, for all $k \geq 1$, we have

1. $g_k^\ell = 8g_{k-1}^\ell + 1$ *and* $g_k^r = 8g_{k-1}^r + 1$ *with* $g_k^\ell = g_k^r$
2. $g_k = 8g_{k-1} + 2$ *and* $g_k = 2g_k^\ell$

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Corollary

Let $k \geq 1$. Then $g_k^\ell = g_k^r = \frac{8^{k-1}-1}{7}$.

Distribution of minimal fractions at a given level

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Lemma

Let $W_i = [0, 1, \omega_i, b_4, \ldots, b_m]$ *for* $0 \leq i \leq 3$ *be continued fractions. Let* N_i *and* D_i *denote respectively the numerator and the denominator of* W_i *for* $0 \leq i \leq 3$. *Then,*

1. $N_0 = N_3$ *and* $D_0 + D_3 = 3N_0$. *In particular* $\frac{N_0}{D_0} \oplus \frac{N_3}{D_3} = \frac{2}{3}$.
2. $N_1 = N_2$ *and* $D_1 + D_2 = 3N_1$. *In particular* $\frac{N_1}{D_1} \oplus \frac{N_2}{D_2} = \frac{2}{3}$.

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- ex: $\frac{7}{12} \oplus \frac{7}{9} = \frac{2}{3} = \frac{8}{13} \oplus \frac{8}{11}$.

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Theorem (Tarsissi,S,N, 2025+)

If $\frac{a}{b}$ is a minimal fraction at level k then there exists a minimal fraction $\frac{a}{b'}$ at level k with $\frac{a}{b} \oplus \frac{a}{b'} = \frac{2}{3}$. Furthermore, $\frac{a}{b}$ and $\frac{a}{b'}$ are equidistant from the extremities of $SB_{/3}[k]$.

Continuous fractions of boundary paths

Definition (Boundary paths)

The minimal boundary path $\mathcal{P}^{\ell}$ (resp. $\mathcal{P}^{c\ell}$, $\mathcal{P}^{cr}$, $\mathcal{P}^{r}$) is defined as the sequence of fractions $\left(f_k^{\ell}\right)_{k\geq 1}$ (resp. $\left(f_k^{c\ell}\right)_{k\geq 1}$, $\left(f_k^{cr}\right)_{k\geq 1}$, $\left(f_k^{r}\right)_{k\geq 1}$).

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Proposition

Let $P(\Omega)$ be a concatenation of elements of Ω and let $\frac{a}{b} = [0, 1, P(\Omega), 2] \in SB_{/3}$ be a minimal fraction appearing at level k on some boundary path $\mathcal{P}$. Then the fraction with continued fraction $[0, 1, P(\Omega), \omega_0, 2]$ is minimal and appears on $\mathcal{P}$ at level $k+1$.

Irrational limits

Theorem (Tarsissi,S,N, 2025+)

For all $k \geq 2$, the continued fraction representations of

$$\begin{array}{lll} f_k^{\ell} & \textit{is} & [0,1,\omega_0,\omega_0^{k-2},2] \\ f_k^{c\ell} & \textit{is} & [0,1,\omega_1,\omega_0^{k-2},2] \\ f_k^{cr} & \textit{is} & [0,1,\omega_2,\omega_0^{k-2},2] \\ f_k^{r} & \textit{is} & [0,1,\omega_3,\omega_0^{k-2},2] \end{array}$$

Furthermore,

$$\begin{array}{lll} \mathcal{P}^{\ell} & \to & \frac{1}{\sqrt{3}} \\ \mathcal{P}^{c\ell} & \to & \frac{16-\sqrt{3}}{23} \\ \mathcal{P}^{cr} & \to & \frac{17+\sqrt{3}}{26} \\ \mathcal{P}^{r} & \to & \frac{3+\sqrt{3}}{6} \end{array}$$

Conclusion+Perspectives

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Perspectives:

1. Determine the irrational limits of the other minimal paths
2. From a geometrical perspective, we have defined a new family of Christoffel words that deserves to be investigated.
3. If we fix a certain value n in SB, where $n = a + b$ for a certain fraction $\frac{a}{b}$, can we find some other characteristics using δ^2.

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Thank you