On sufficient conditions for the linear independence of infinite series with exponents n^k

Takafumi Tsurumaki (University of Tsukuba)

Numeration and Substitution 2025

Self Introduction

- Name: Takafumi Tsurumaki
- Affiliation: University of Tsukuba, first-year Ph.D. student
- Supervisior: Hajime Kaneko
- Field: transcendental number theory, uniformly distribution
- Hometown: Yamagata Prefecture in Japan
- Interests: tiling, Diophantine approximation, circle method e.t.c....
- Hobbies: thinking logical puzzles, watching anime e.t.c....

Definition of $\gamma_b(k)$

For $b, k \in \mathbb{Z}_{\geq 2}$, we define $\gamma_b(k) := \displaystyle\sum_{n=0}^{\infty} \frac{1}{b^{n^k}} \notin \mathbb{Q}$.

Example

In the case $b = 2$:

$$\gamma_2(2) = (\overset{0}{1}.\overset{1}{1}00\overset{4}{1}0000\overset{9}{1}000000\overset{16}{1}00000000\overset{25}{1}000\ldots)_2,$$

$$\gamma_2(3) = (\overset{0}{1}.\overset{1}{1}000000\overset{8}{1}000000000000000000\overset{27}{1}0\ldots)_2,$$

$$\gamma_2(4) = (\overset{0}{1}.\overset{1}{1}00000000000000\overset{16}{1}000000000000\ldots)_2.$$

Open problems

$b \in \mathbb{Z}_{\geq 2}$: fixed. $k \in \mathbb{Z}_{\geq 2}$. $\gamma_b(k) = \sum_{n=0}^{\infty} \frac{1}{b^{n^k}} \notin \mathbb{Q}$.

Problem 1

$k \in \mathbb{Z}_{\geq 2}$, $\gamma_b(k)$: a transcendental number?

Problem 2

$k, \ell \in \mathbb{Z}_{\geq 2}$, $k < \ell$. $\gamma_b(k), \gamma_b(\ell)$: algebraically independent?

i.e.) $\{\gamma_b(k)^i \gamma_b(\ell)^j \mid (i, j) \in (\mathbb{Z}_{\geq 0})^2\}$: linearly independent over $\mathbb{Q}$?

Main results

$$\gamma_b(k) = \sum_{n=0}^{\infty} \frac{1}{b^{n^k}}.$$

Theorem

We obtained $B \subset (\mathbb{Z}_{\geq 0})^2$ *satisfying the following:*

$$\left\{\gamma_b(k)^i \gamma_b(\ell)^j \mid (i, j) \in B\right\} \text{ is linearly independent over } \mathbb{Q}.$$

We obtained different types of B*. For details, please see my poster!*

Example

The set $\left\{\gamma_b(5)^i \gamma_b(23)^j \mid (i, j) \in (\mathbb{Z}_{\geq 0})^2, 0 \leq i + j \leq 3\right\}$ **(# of elements = 10)** is linearly independent over $\mathbb{Q}$.

Background

$$\gamma_b(k) = \sum_{n=0}^{\infty} \frac{1}{b^{n^k}}.$$

- Special case of P. Erdős's result (1957):

Partial results for transcendence

$\gamma_b(k)$: either a transcendental number or an algebraic number of degree at least k.

- Special case of S. Murakami and Y. Tachiya's result (2024):

Linear independence of the numbers with degree at most 1

$$1, \sum_{n=1}^{\infty} \frac{1}{b^{in^j}} \quad (i = 1, 2, \cdots, \ j = 2, 3, \cdots) \text{ are linearly independent over } \mathbb{Q}.$$