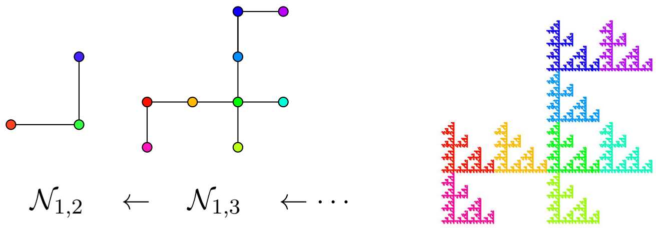
On the topology of the limit set of non-autonomous IFS

Takayuki Watanabe (Chubu University) ✉ takawatanabe@fsc.chubu.ac.jp
with Yuto Nakajima (Doshisha University) ✉ yunakaji@mail.doshisha.ac.jp

Numeration and Substitution 2025 @Tsukuba, 8–12 Sep.

Fractal meets Algebraic Topology

We extract the essence of the celebrated Hata theorem and use **simplicial complexes** to study the shape of fractals.



Definition of non-autonomous IFS

We define a non-autonomous IFS as follows.

- Let X be a compact metric space.
- Let $\Phi^{(j)} = \{f_i^{(j)} : X \to X\}_{i \in I^{(j)}}$ be a collection of maps with $\#I^{(j)} < \infty$ for every $j \geq 1$.
- Assume that $\exists c < 1$ such that $\mathrm{Lip}(f_i^{(j)}) \leq c$ for each $j \geq 1$ and for each $i \in I^{(j)}$.

For $(\Phi^{(j)})_{j=1}^{\infty}$, define the limit set by

$$J := \bigcap_{j=1}^{\infty} \bigcup_{(i_1, i_2, \dots i_j) \in \prod_{\ell=1}^{j} I^{(\ell)}} f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \cdots \circ f_{i_j}^{(j)}(X).$$

Self-similarity

For every $k > 1$, consider $(\Phi^{(j-1+k)})_{j=1}^{\infty}$ and denote its limit set by J_k. Then we have $J = J_1$, and

$$J_1 = \bigcup_{(i_1, \dots, i_{k-1}) \in \prod_{\ell=1}^{k-1} I^{(\ell)}} f_{i_1}^{(1)} \circ \cdots \circ f_{i_{k-1}}^{(k-1)}(J_k).$$

Nerves of non-autonomous IFS

For every $k > 1$, define the simplicial complex $\mathcal{N}_{1,k}$ as the nerve of the covering:

- Regard each $v = (i_1, \dots, i_{k-1}) \in \prod_{\ell=1}^{k-1} I^{(\ell)}$ as a vertex of $\mathcal{N}_{1,k}$, and define $f_v = f_{i_1}^{(1)} \circ \cdots \circ f_{i_{k-1}}^{(k-1)}$.
- A set $\{v_0, v_1, \dots, v_q\}$ is a q-simplex of $\mathcal{N}_{1,k}$ if and only if $\bigcap_{p=0}^{q} f_{v_p}(J_k) \neq \emptyset$.

Define a simplicial map $\phi_k \colon \mathcal{N}_{1,k+1} \to \mathcal{N}_{1,k}$ so that

$$\phi_k(i_1, \dots, i_{k-1}, i_k) = (i_1, \dots, i_{k-1}).$$

It induces a hom. $(\phi_k)_* \colon H_*(\mathcal{N}_{1,k+1}) \to H_*(\mathcal{N}_{1,k})$ on the homology groups. We call the inverse limit $\varprojlim_k H_q(\mathcal{N}_{1,k})$ the qth **Čech–Sumi homology group**.

Theorem 1. We have $\check{H}_*(J) \underset{\text{isom}}{\cong} \varprojlim H_*(\mathcal{N}_{1,k})$ between the Čech and the Čech–Sumi homology group. More precisely regarding connectedness, TFAE.

- The limit set J is connected.
- The limit set J is path-connected.
- For every $k > 1$, the nerve $\mathcal{N}_{1,k}$ is connected.

Moreover, the limit set J is totally disconnected if

$$\lim_{k \to \infty} c^{k-1} \cdot (\max \# \text{ of vertices in a comp. of } \mathcal{N}_{1,k}) = 0.$$

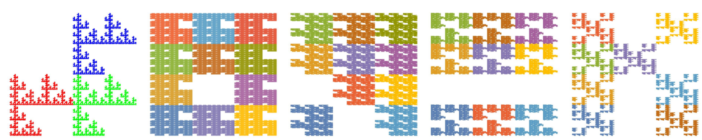
For simple examples, we can calculate the nerves.

Example: fractal square

Let $n_1, n_2 \in \mathbb{N}_{\geq 2}$ and set $I = \prod_{k=1}^{2} \{0, 1, \dots, n_k - 1\}$. For each $\mathbf{i} = (i_1, i_2) \in I$, define a contractive map by

$$f_{\mathbf{i}} : [0,1]^2 \ni (x_1, x_2) \mapsto \left(\frac{x_1 + i_1}{n_1}, \frac{x_2 + i_2}{n_2} \right) \in [0,1]^2 = X.$$

Fix r, and choose each $I^{(j)}$ so that $\#(I \setminus I^{(j)}) = r$ uniformly **at random**. Then $\Phi^{(j)} = \{f_{\mathbf{i}}\}_{\mathbf{i} \in I^{(j)}}$ forms a non-autonomous IFS. The limit sets J are depicted below.



Theorem 2. For a.e. non-auto. fractal square J, we have $\check{H}_q(J) \cong 0$ for every $q \geq 2$. Moreover:

(i) If $r = 1$, then J is connected and locally conn.
  (a) If $n_1 = n_2 = 2$, then $\check{H}_1(J) \cong 0$.
  (b) If $(n_1, n_2) \neq (2, 2)$, then
  $$\lim_{k \to \infty} \frac{1}{k} \log(\mathrm{rank} H_1(\mathcal{N}_{1,k})) = \log(n_1 n_2 - r).$$

(ii) If $2 \leq r < \min\{n_1, n_2\}$, then $\mathrm{rank}\check{H}_0(J) = \infty$, $\mathrm{rank}\check{H}_1(J) = \infty$, and
$$\lim_{k \to \infty} \frac{1}{k} \log(\mathrm{rank} H_1(\mathcal{N}_{1,k}) - \mathrm{rank} H_0(\mathcal{N}_{1,k})) = \log(n_1 n_2 - r).$$

(iii) If $n_1 \leq r < n_2$ (resp. $n_2 \leq r < n_1$), then every connected component of J is horizontal (resp. vertical) line segment. One of them is a line of length 1 and others may degenerate to a singleton.

(iv) If $r \geq \max\{n_1, n_2\}$, then J is totally disconnected.