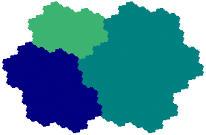
Low symbolic discrepancy in the Tribonacci shift

Maria Clara Werneck

IRIF, U. Paris Cité & LAMFA, U. de Picardie Jules Verne

Numeration, Tsukuba, Japan

September 8th, 2025



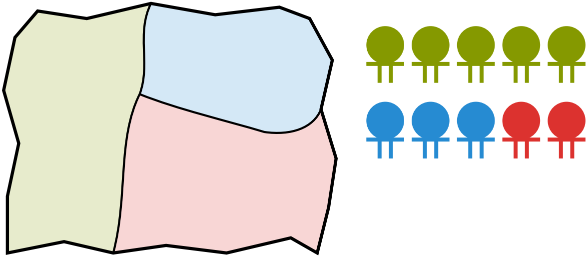
Co-funded by the
European Union

Outline

1. Introduction
2. Literature
3. Results: Tribonacci Shift (and Fibonacci)
4. Conclusion

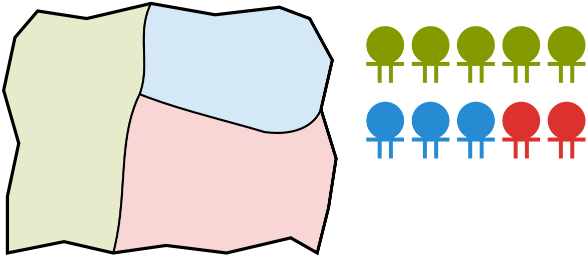
Introduction

The chairperson assignment problem



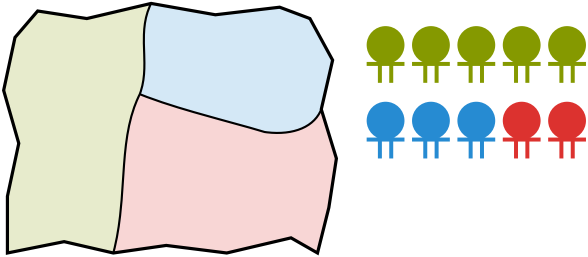
- d states form a congress
- Each year a president for the congress is selected

The chairperson assignment problem



- d states form a congress
- Each year a president for the congress is selected
- Goal: assign the president fairly

The chairperson assignment problem



- d states form a congress
- Each year a president for the congress is selected
- Goal: assign the president fairly
- At any time, for each state, the accumulated number of presidents from that state is proportional to its weight

Let $\mathcal{A} = \{\texttt{0,1, ..., d-1}\}$ be an alphabet and endow $\mathcal{A}^{\mathbb{N}}$ with the product topology.

Let $\mathcal{A} = \{\texttt{0,1, ..., d-1}\}$ be an alphabet and endow $\mathcal{A}^{\mathbb{N}}$ with the product topology.

Take a sequence/word $u = (u_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ to represent from where is the president on the year n.

Let $\mathcal{A} = \{\texttt{0,1, ..., d-1}\}$ be an alphabet and endow $\mathcal{A}^{\mathbb{N}}$ with the product topology.

Take a sequence/word $u = (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ to represent from where is the president on the year n.

A frequency vector $\vec{\alpha} \in \mathbb{R}^d$ is such that

$$\sum_{a \in \mathcal{A}} \alpha_a = 1 \text{ and, for every } a \in \mathcal{A},\ \alpha_a \geq 0.$$

The discrepancy of a word u given a frequency vector $\vec{\alpha}$ is:

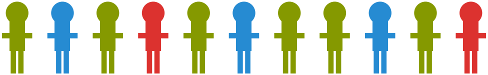
$$\Delta_{\vec{\alpha}}(u) := \max_{a \in \mathcal{A}} \sup_{n \in \mathbb{N}} \left|\Delta^a(u_0 u_1 \cdots u_{n-1})\right|,$$

where, for every letter a,

$$\Delta^a(u_0 u_1 \cdots u_{n-1}) := \#\{0 \leq i < n : u_i = a\} - n\alpha_a.$$

An example

States	Weights
Green	0.5
Blue	0.3
Red	0.2



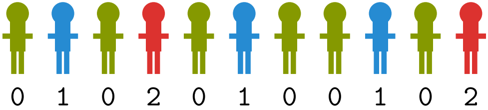
0 1 0 2 0 1 0 0 1 0 2

$$\begin{cases} \Delta^0(0) = 1 - 0.5 = 0.5 \\ \Delta^1(0) = 0 - 0.3 = -0.3 \\ \Delta^2(0) = 0 - 0.2 = -0.2 \end{cases}$$

$$\max_{a \in \mathcal{A}} |\Delta^a(0)| = 0.5$$

An example

States	Weights
Green	0.5
Blue	0.3
Red	0.2

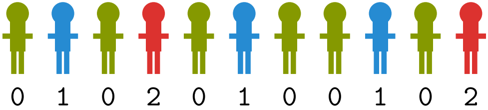


$$\begin{cases} \Delta^0(01) = 1 - 2 \cdot 0.5 = 0.0 \\ \Delta^1(01) = 1 - 2 \cdot 0.3 = 0.4 \\ \Delta^2(01) = 0 - 2 \cdot 0.2 = -0.4 \end{cases}$$

$$\max_{n \in \{1,2\}} \max_{a \in \mathcal{A}} \left| \Delta^a(u_{[0,n)}) \right| = 0.5$$

An example

States	Weights
Green	0.5
Blue	0.3
Red	0.2

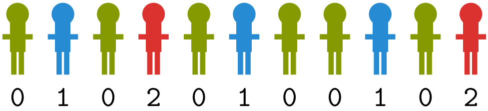


$$\begin{cases} \Delta^0(010) = 2 - 3 \cdot 0.5 = 0.5 \\ \Delta^1(010) = 1 - 3 \cdot 0.3 = 0.1 \\ \Delta^2(010) = 0 - 3 \cdot 0.2 = -0.6 \end{cases}$$

$$\max_{n \in \{1,2,3\}} \max_{a \in \mathcal{A}} \left|\Delta^a(u_{[0,n)})\right| = 0.6$$

An example

States	Weights
Green	0.5
Blue	0.3
Red	0.2



$$\Delta_{\vec{\alpha}}(u) := \max_{a \in \mathcal{A}} \sup_{n \in \mathbb{N}} |\Delta_a(n)| \geq 0.6$$

Quick pause: Symbolic dynamics notation

The abelianization or Parikh vector of a finite word w is

$$\vec{l}(w) = (|w|_1, |w|_2, \ldots, |w|_d),$$

where $|w|_i$ denotes the number of times the letter i occurs in w.

A factor of an infinite word u is a finite word denoted by $u_{[i,j)}$, $i < j$, with

$$u_{[i,j)} = u_i u_{i+1} \cdots u_{j-1}.$$

The frequency of a letter a in an infinite word u (if the limit exists) is:

$$\alpha_a = \lim_{n\to\infty} \frac{\#\{0 \le i < n : u_i = a\}}{n}.$$

Literature

Given a word u, what is the asymptotic behaviour of its discrepancy? Is it bounded?

Given a word u, what is the asymptotic behaviour of its discrepancy? Is it bounded?

Theorem. (B. Adamczewski'03,'04) The discrepancy is bounded in a substitution word of Pisot type (algebraic condition).

Questions

What can we say about low discrepancy?

What can we say about low discrepancy?

Define

$$D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u),$$

where u is a word over d letters.

What can we say about low discrepancy?

Define

$$D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u),$$

where u is a word over d letters.

Is D_d bounded?

Can we bound $D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u)$?

- Niederreiter'72: $D_d \leq d - 1$

Historical Development

Can we bound $D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u)$?

- Niederreiter'72: $D_d \leq d - 1$
- Tijdeman'73: $D_d \leq 1 - \dfrac{1}{2d - 2}$

Historical Development

Can we bound $D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u)$?

- Niederreiter'72: $D_d \leq d - 1$
- Tijdeman'73: $D_d \leq 1 - \dfrac{1}{2d-2}$
- Meijer'73: $D_d = 1 - \dfrac{1}{2d-2}$

Can we bound $D_d = \sup_{\vec{\alpha}} \inf_{u \in \mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u)$?

- Niederreiter'72: $D_d \leq d - 1$
- Tijdeman'73: $D_d \leq 1 - \frac{1}{2d-2}$
- Meijer'73: $D_d = 1 - \frac{1}{2d-2}$
- Schneider'96: [$\vec{\alpha}$ coordinates are $\mathbb{Q}$−linearly independent]

$$\Delta_{\vec{\alpha}}(u) \geq 1 - \frac{1}{d}$$

Main question

What are the shift dynamical systems (X_u, S) that contain an infinite word $v \in X_u$ that attains low discrepancy?

Main question

What are the shift dynamical systems (X_u, S) that contain an infinite word $v \in X_u$ that attains low discrepancy?

$\rightarrow X_u = \overline{\bigcup_{k\in\mathbb{N}} S^k(u)} \subset \mathcal{A}^{\mathbb{N}}$

Main question

What are the shift dynamical systems (X_u, S) that contain an infinite word $v \in X_u$ that attains low discrepancy?

$\rightarrow X_u = \overline{\bigcup_{k\in\mathbb{N}} S^k(u)} \subset \mathcal{A}^{\mathbb{N}}$

$\rightarrow \exists\, v \in X_u,$

$$\Delta_{\vec{\alpha}}(v) \leq 1 - \frac{1}{2d-2} = D_d = \sup_{\vec{\alpha}} \inf_{u\in\mathcal{A}^{\mathbb{N}}} \Delta_{\vec{\alpha}}(u)$$

Results: Tribonacci Shift

Tribonacci word

Tribonacci word u is $\lim_{n\to+\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

Tribonacci word u is $\lim_{n\to+\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

0

Tribonacci word

Tribonacci word u is $\lim_{n\to+\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01$$

Tribonacci word

Tribonacci word u is $\lim_{n\to+\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01 \to 0102$$

Tribonacci word

Tribonacci word u is $\lim_{n \to +\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01 \to 0102 \to 0102010$$

Tribonacci word u is $\lim_{n\to+\infty} \sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01 \to 0102 \to 0102010 \to \cdots$$

Tribonacci word

Tribonacci word u is $\lim_{n\to+\infty}\sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01 \to 0102 \to 0102010 \to \cdots$$

$$u = 0102010010201\cdots$$

Tribonacci word u is $\lim_{n\to+\infty}\sigma^n(0)$:

$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$$

$$0 \to 01 \to 0102 \to 0102010 \to \cdots$$

$$u = 0102010010201\cdots$$

Is it true for the Tribonacci shift, i.e, is there $v \in X_u$ where $\Delta_{\vec{\alpha}}(v) < \frac{3}{4}$?

Theorem (in preparation)

There exists a word w that belongs to the Tribonacci shift that achieves low discrepancy, that is, $\Delta_{\vec{\alpha}}(w) < \frac{3}{4}$.
The word w can be explicitly described as:

$$w = \lim_{k \to +\infty} \mathit{01}\sigma(1)\sigma^2(\mathit{2}) \cdots \sigma^{4k}(\mathit{1})\sigma^{4k+1}(\mathit{1})\sigma^{4k+2}(\mathit{2}).$$

The balancedness constant B_u of a word u is defined as:

$$\inf\{B > 0 \mid \forall v, w \text{ factors of } u, |v| = |w|, \forall a \in \mathcal{A},\ ||v|_a - |w|_a| \leq B\}$$

The balancedness constant B_u of a word u is defined as:

$$\inf\{B > 0 \mid \forall v, w \text{ factors of } u, |v| = |w|, \forall a \in \mathcal{A},\ ||v|_a - |w|_a| \leq B\}$$

Fact (modified B. Adamczweski'03)

For a minimal shift (X_u, S),

$$\sup_{v \in X_u} \Delta_{\vec{\alpha}}(v) \leq B_u \leq 4 \inf_{v \in X_u} \Delta_{\vec{\alpha}}(v)$$

Corollary (G. Richomme, K. Saari, L. Zamboni'10)

The Tribonacci word is 2-balanced.

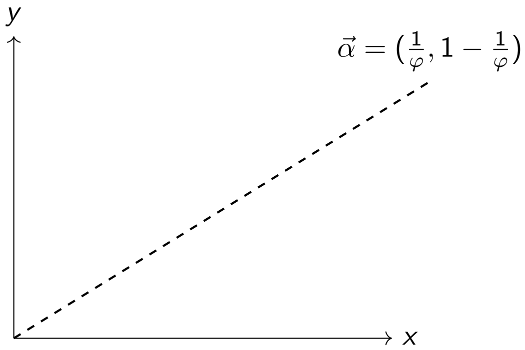
Corollary (G. Richomme, K. Saari, L. Zamboni'10)

The Tribonacci word is 2-balanced.

Proof.

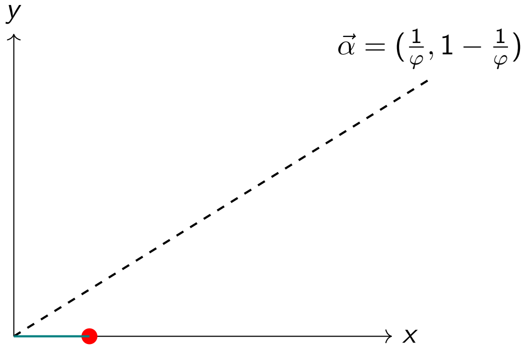
$B_u < 4 \inf_{v \in X_u} \Delta_{\vec{\alpha}}(v) < 4 \cdot \frac{3}{4}$ and $|101|_1 - |020|_1 = 2$. $\square$

Fibonacci word $u = 01001010010010100\cdots$, golden ratio φ



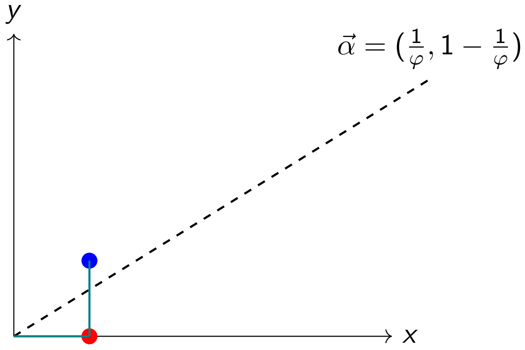
Viewing discrepancy geometrically

Fibonacci word $u = 01001010010010100\cdots$, golden ratio φ



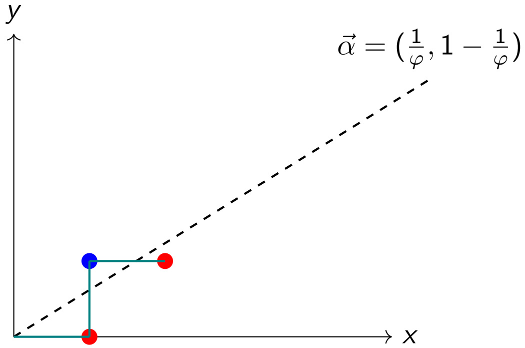
0

Fibonacci word $u = 01001010010010100\cdots$, golden ratio φ



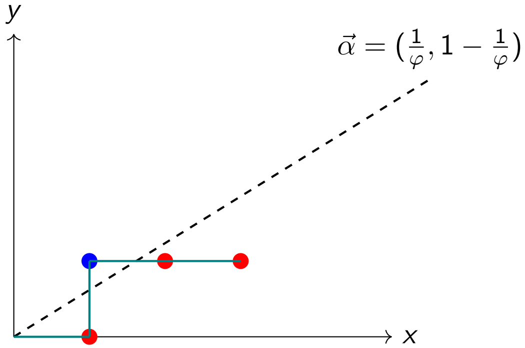
01

Fibonacci word $u = 0100101001001010 0\cdots$, golden ratio φ



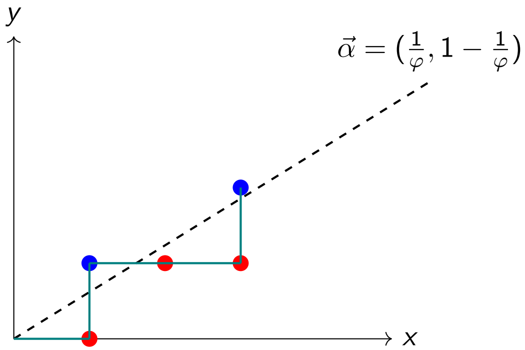
010

Fibonacci word $u = 01001010010010100\cdots$, golden ratio φ



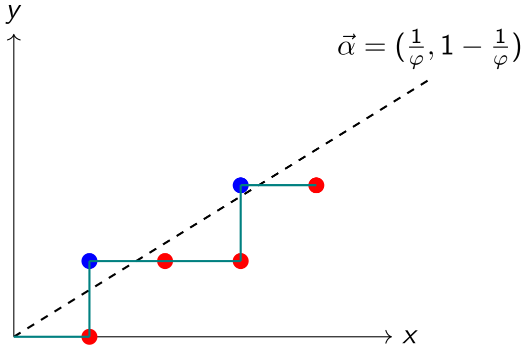
0100

Fibonacci word $u = 010010100100101 00\cdots$, golden ratio φ



01001

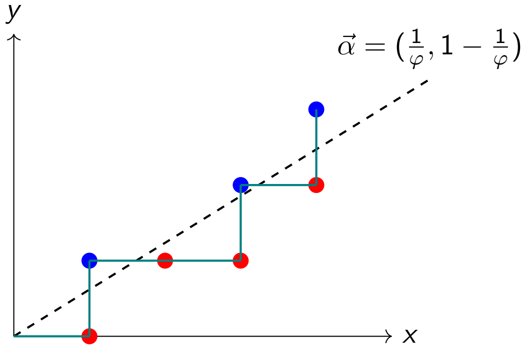
Fibonacci word $u = 0100101001001010 0\cdots$, golden ratio φ



010010

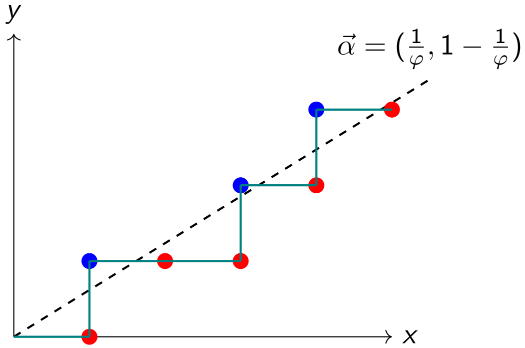
Viewing discrepancy geometrically

Fibonacci word $u = 0100101001001010 0\cdots$, golden ratio φ



0100101

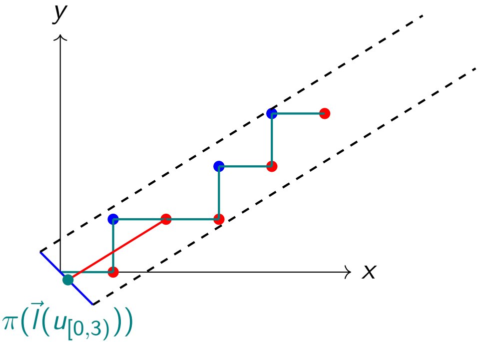
Fibonacci word $u = 0100101001001010 0\cdots$, golden ratio φ



01001010

Each point is represented by the abelianization $\vec{l}(u_{[0,n)})$

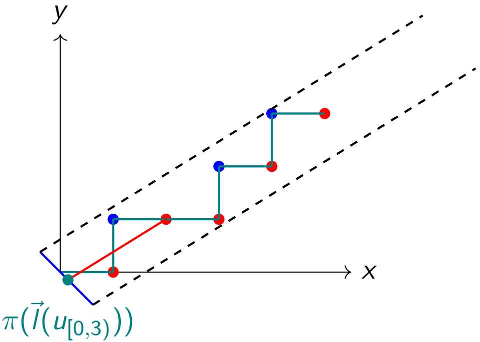
Project along $\vec{\alpha}$ in $\mathbf{1}^{\perp}$: $\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n)})) = \vec{l}(u_{[0,n)}) - n\vec{\alpha}$



Viewing discrepancy geometrically

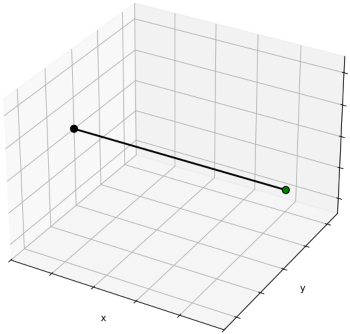
Each point is represented by the abelianization $\vec{l}(u_{[0,n)})$

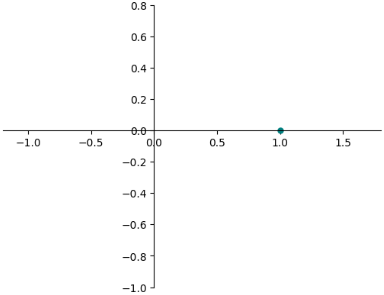
Project along $\vec{\alpha}$ in $\mathbf{1}^{\perp}$: $\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n)})) = \vec{l}(u_{[0,n)}) - n\vec{\alpha}$



(Discrepancy) $\Delta_{\vec{\alpha}}(u) = \sup_{n\in\mathbb{N}} \|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n)}))\|_{\infty}$

Geometry of the discrepancy of the Tribonacci word u:

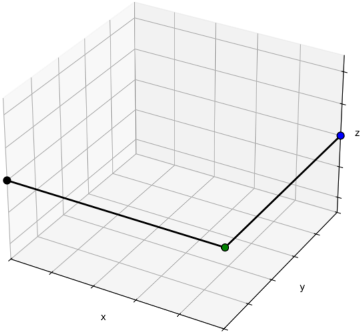


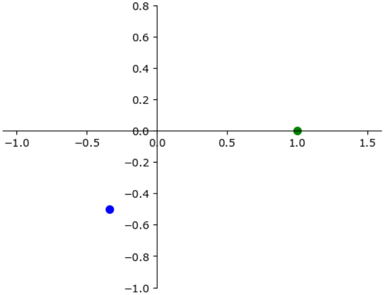


$$u = 0 \cdots$$

Tribonacci case: Rauzy Fractal

Geometry of the discrepancy of the Tribonacci word u:

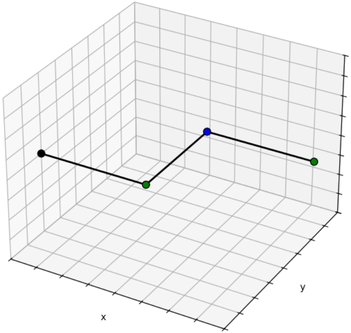


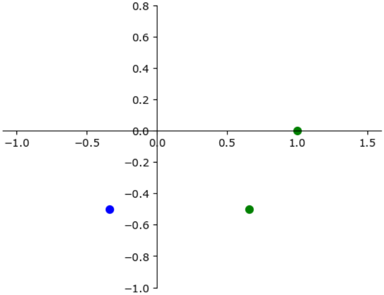


$$u = 01 \cdots$$

Tribonacci case: Rauzy Fractal

Geometry of the discrepancy of the Tribonacci word u:

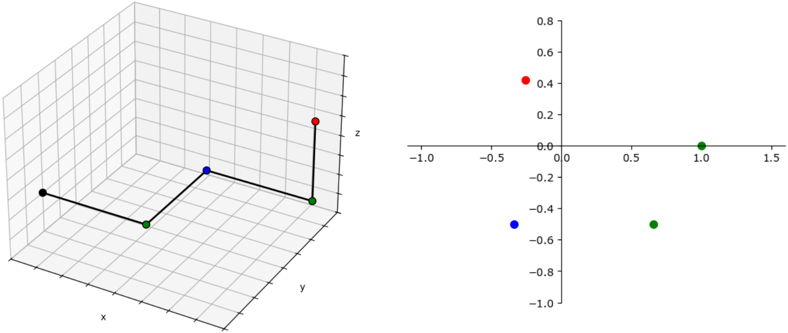




$$u = \color{green}{0}\color{blue}{1}\color{green}{0}\cdots$$

Tribonacci case: Rauzy Fractal

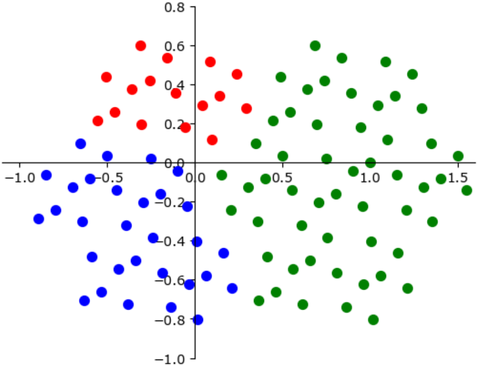
Geometry of the discrepancy of the Tribonacci word u:



$$u = 0102\cdots$$

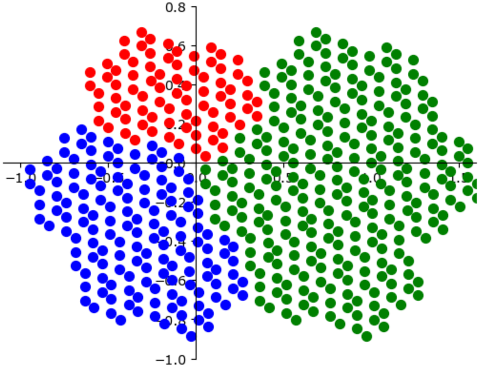
Tribonacci case: Rauzy Fractal

Geometry of the discrepancy of the Tribonacci word u:



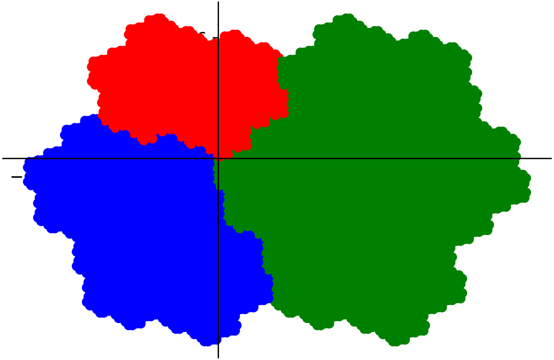
Tribonacci case: Rauzy Fractal

Geometry of the discrepancy of the Tribonacci word u:

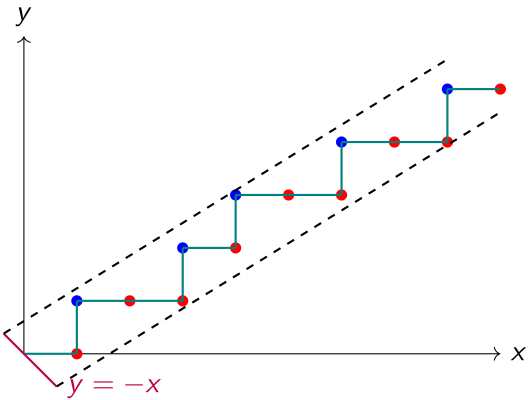


Tribonacci case: Rauzy Fractal

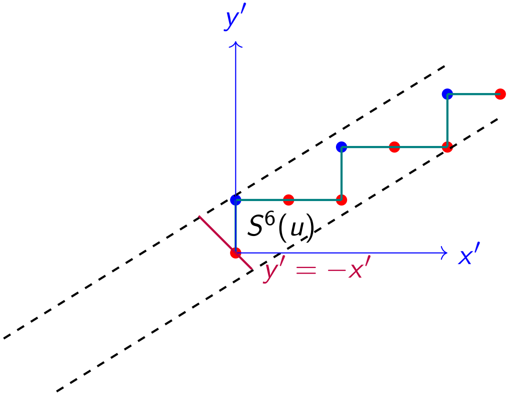
Geometry of the discrepancy of the Tribonacci word u:



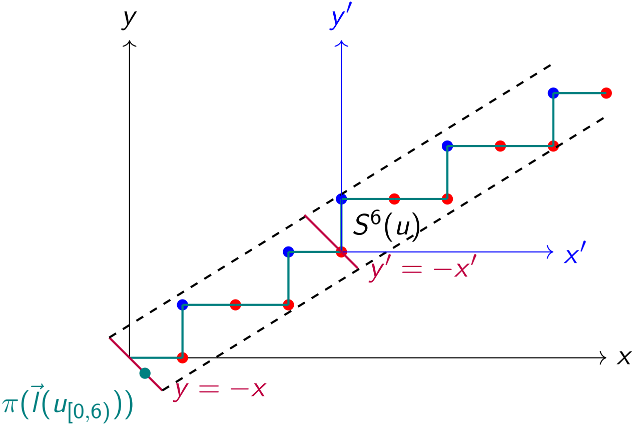
What happens with the discrepancy when we shift u?



What happens with the discrepancy when we shift u?



What happens with the discrepancy when we shift u?



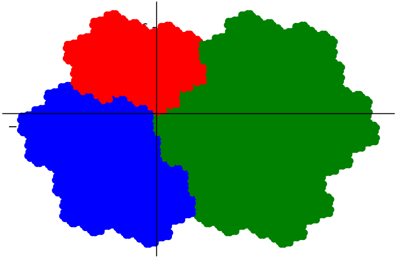
Proposition

Let u be an infinite word. Consider the dynamical system (X_u, S) generated by u. Let $\vec{\alpha}$ be a frequency vector. Then, for every $k \in \mathbb{N}$,

$$\Delta_{\vec{\alpha}}(S^k(u)) = \sup_{n \in \mathbb{N}} \left| \|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,n+k)}))\|_\infty - \|\pi_{\vec{\alpha}}(\vec{l}(u_{[0,k)}))\|_\infty \right|.$$

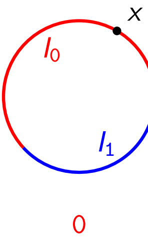
Main idea

- Understanding the discrepancy with the Rauzy fractal
- Encode the symmetrical point of the Rauzy Tribonacci fractal
- Prefix-suffix automaton
- (1989) Dumont-Thomas decomposition
- (2001) Canterini-Siegel decomposition



Results: Fibonacci Shift

Fibonacci case: rotation

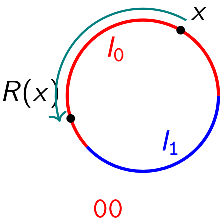


$$R_\varphi : [\varphi - 1, 2 - \varphi) \to [\varphi - 1, 2 - \varphi)$$
$$x \to x + \varphi \mod 1$$

Orbit of x under R_φ coded by the word $w \in \{0,1\}^{\mathbb{Z}}$

$$w_i = a \iff R^i_\varphi(x) \in I_a$$

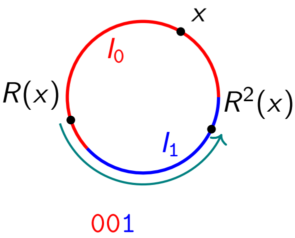
Fibonacci case: rotation



$$R_\varphi : [\varphi - 1, 2 - \varphi) \to [\varphi - 1, 2 - \varphi)$$
$$x \to x + \varphi \mod 1$$

Orbit of x under R_φ coded by the word $w \in \{0,1\}^{\mathbb{Z}}$

$$w_i = a \iff R_\varphi^i(x) \in I_a$$

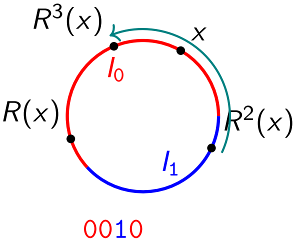


$$R_\varphi : [\varphi - 1, 2 - \varphi) \to [\varphi - 1, 2 - \varphi)$$
$$x \to x + \varphi \mod 1$$

Orbit of x under R_φ coded by the word $w \in \{0,1\}^{\mathbb{Z}}$

$$w_i = a \iff R^i_\varphi(x) \in I_a$$

Fibonacci case: rotation



$$R_\varphi : [\varphi - 1, 2 - \varphi) \to [\varphi - 1, 2 - \varphi)$$
$$x \to x + \varphi \quad \text{mod } 1$$

Orbit of x under R_φ coded by the word $w \in \{0,1\}^{\mathbb{Z}}$

$$w_i = a \iff R^i_\varphi(x) \in I_a$$

Complete understanding of discrepancy of every word in the shift:

Theorem (in preparation)

Let w be an infinite word in the Fibonacci shift encoded by the of orbit of x under the rotation R_φ. Its discrepancy satisfies:

$$\Delta_{\vec{\alpha}}(w) = \max(|\varphi - 1 - x|, |\varphi - x|).$$

Corollary

There exists an infinite word w in the Fibonacci shift that atains minimum discrepancy, that is, $\Delta_{\vec{\alpha}}(w) = \frac{1}{2}$. The word w can be described as:

$$w = \lim_{k \to +\infty} \mathit{01}\sigma^2(\mathit{0})\sigma^5(\mathit{0}) \ldots \sigma^{3k+2}(\mathit{0}).$$

Conclusion

Research direction

- Values of low discrepancy for m-bonacci shifts
- For which other classes of substitutions can we find words in the shift which achieve low discrepancy, that is:

$$\exists\, v \in X_u, \quad \Delta_{\vec{\alpha}}(v) \leq 1 - \frac{1}{2d-2},$$

  where d is the number of letters?
- Do subshifts that achieve low discrepancy have other common (and nice) spectral properties, as pure discrete spectrum?

Research direction

- Values of low discrepancy for m-bonacci shifts
- For which other classes of substitutions can we find words in the shift which achieve low discrepancy, that is:

$$\exists\, v \in X_u, \quad \Delta_{\vec{\alpha}}(v) \leq 1 - \frac{1}{2d-2},$$

  where d is the number of letters?
- Do subshifts that achieve low discrepancy have other common (and nice) spectral properties, as pure discrete spectrum?

Thank you!