Numeration and Substitution

September 8 - 12, 2025

University of Tsukuba

Conference Schedule

	Monday	Tuesday	Wednesday	Thursday	Friday
	Registration 09:00-10:00				
10:00-11:00	**Kota Saito**	**Pieter Allaart**	**Seonhee Lim**	**Baowei Wang**	**Mumtaz Hussain**
11:00-11:30	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
11:30-12:00	**Makoto Kawashima**	**Derong Kong**	**Dominik Kwietniak**	**Mélodie Andrieu**	**Yao-Qiang Li**
12:00-12:30	**Seul Bee Lee**	**Savinien Kreczman**	**Jorge Olivares-Vinales**	**Anjelo Gabriel Cruz**	**Hiroki Takahasi**
12:30-14:15	Lunch	Lunch		Lunch	
14:15-14:45	**Manon Stipulanti**	**Jun Luo**		**Nathaniel Nollen**	
14:45-15:15	**Pitch Talk (1)**	**Kiko Kawamura**		**Lama Tarsissi**	
15:15-15:30	Coffee Break	Coffee Break		Coffee Break	
15:30-16:00	**Maria Clara Werneck**	**Stefano Marmi**		**Johannes Jaerisch**	
16:00-16:30	**Toghrul Karimov**	**Pitch Talk (2)**		**Hiromi Ei**	
16:30-17:00	**Antoine Renard**	**Poster** 16:30-18:00			
		Dinner 18:30-21:00			

Invited talks

Pieter Allaart

University of North Texas

Title The β-transformation with a hole at 0

Abstract The β-transformation $T_\beta(x) := \beta x \bmod 1$, where $\beta > 1$ is a real constant, has been studied by many authors due to its rich spectrum of dynamical properties. In 2018, Kalle, Kong, Langeveld and Li considered the β-transformation with $\beta \in (1, 2]$ and a hole $(0, t)$, and studied the survivor set $K_\beta(t)$ of points whose forward orbit never enters the hole. Extending a well-known result of Urbański (1987), they showed that for fixed β, the Hausdorff dimension of $K_\beta(t)$ is a decreasing devil's staircase as a function of t, but left open the question of where this function reaches the value 0. In recent work with D. Kong, we gave a complete answer to this question for all $\beta > 1$, which I will describe in this talk. The methods involve combinatorics on words, particularly Farey words, and subshifts determined by pairs of lexicographical inequalities. I will also discuss the relative sizes and local dimension of two bifurcation sets: one for the set-valued function $t \mapsto K_\beta(t)$ and one for the function $t \mapsto \dim_H K_\beta(t)$. (Joint work with D. Kong.)

Mumtaz Hussain

La Trobe University

Title The Generalised Baker-Schmidt Problem

Abstract The Generalised Baker-Schmidt Problem (1970) concerns the f-dimensional Hausdorff measure of the set of well-approximable points on a nondegenerate manifold. There are two variants of this problem: simultaneous and dual approximation. The divergence cases have long been established for dual approximation on arbitrary nondegenerate manifolds. The corresponding convergence case remains a major and challenging open question. In this talk, I will walk through some recent progress for manifolds of co-dimension 1 and 2.

Seonhee Lim

Seoul National University

Title Complex continued fractions and Kleinian circle packings

Abstract Complex continued fraction map shares nice dynamical properties with the classical continued fraction map. Some of these properties can be generalized to a large family of circle packings including geometrically finite Kleinian circle packings. This talk is based on joint work with Jungwon Lee, Dohyeong Kim, and with Kangrae Park and Yongquan Zhang.

Kota Saito
Nihon University

Title Arithmetic properties of prime-representing constants

Abstract In 1947, Mills constructed a real number A such that $[A^{3^k}]$ is always a prime number for every positive integer k. The smallest element of such A's is called Mills' constant. Some previous efforts failed to prove the irrationality of this number. However, the speaker determined that Mills' constant is irrational in the last year. This problem is closely related to the divisibility of powers of a Pisot number. The speaker plans to talk about historical backgrounds, proof, and the recent progress on this problem.

Baowei Wang
Huazhong University of Science and Technology

Title Intrinsic Diophantine approximation on triadic Cantor set

Abstract Recently, Diophantine approximation on fractals has attracted much attention and leaves many interesting unsolved questions. In this talk, I will give a short survey on the recent progress and then a small result about the intrinsic Diophantine approximation on triadic Cantor set K, i.e. approximating real numbers in K by rational numbers in K.

Contributed talks

Mélodie Andrieu

CNRS France - Universitad de Chile

Title A normality conjecture on rational base number systems

Abstract The numeration system in rational base p/q was introduced in 2008 by Akiyama, Frougny, and Sakarovitch. Fixing $p > q$ coprime positive integers, every integer n admits a unique finite expansion in the rational base p/q, which is a finite word over the alphabet $\{0, ..., p-1\}$. Given such an expansion u, we are interested in the minimal words of seed u, namely those finite words w that are lexicographically minimal in their length among the words v for which the concatenation uv is still the expansion of an integer in rational base p/q. As the length increases, the sequence of these minimal words of seed u converges to an infinite word $\mathrm{wmin}(u)$, which is a priori written over the alphabet $\{0, ..., p-1\}$. We conjecture that all such words $\mathrm{wmin}(u)$ are normal over the subalphabet $\{0, ..., q-1\}$. The aim of the talk is to convince the audience that this conjecture seems true and considerably difficult.

Anjelo Gabriel Cruz

Institute of Mathematics, UP Diliman

Title Attractors of Digit Systems Corresponding to Expanding Rational Companion Matrices

Abstract Let A be the 2×2 companion matrix of an irreducible polynomial of the form $p(x) = x^2 + \alpha x + \beta \in \mathbb{Q}[x]$ and $\mathbb{Z}^2[A]$ be the smallest A-invariant $\mathbb{Z}$-module containing $\mathbb{Z}^2$. In this presentation, we discuss the cases to determine the values of α and β for which every vector $x \in \mathbb{Z}^2[A]$ can be written in the form

$$x = d_0 + Ad_1 + \cdots + A^k d_k,$$

where k is a nonnegative integer and $d_0, \ldots, d_k$ are the elements of an appropriately chosen finite subset $\mathcal{D}$ of $\mathbb{Z}^2[A]$ which is a complete residue system of $\mathbb{Z}^2[A]/A\mathbb{Z}^2[A]$, with $d_k \neq 0$. To this end, we introduce finite-state automata which realize the addition of a vector in $\mathbb{Z}^2[A]$ by the vectors $(1, 0)^\top$ and $(0, 1)^\top$ and use induction arguments to establish the existence of an expansion of the above form for any element of $\mathbb{Z}^2[A]$.

Hiromi Ei
Hirosaki University

Title Construction of boundaries of tiles associated with nearest integer complex continued fractions over imaginary quadratic fields

Abstract As natural extensions of nearest-integer complex continued fraction maps over imaginary quadratic fields, we construct tilings whose prototiles appear to have fractal boundaries. H. Nakata, R. Natsui, and I have proved that, in some cases, the boundary of these tiles is a simple closed curve. In this talk, I will explain how to construct the boundaries of these tiles.

Johannes Jaerisch
Nagoya University

Title Convergence rate at infinity for the cusp winding process of Schottky groups

Abstract It is well known that for Lebesgue almost every real number, the arithmetic mean of the continued fraction expansion tends to infinity. It is also known that the Hausdorff dimension spectrum of numbers whose arithmetic mean tends to a given α defines a strictly increasing real-analytic function on $(1, +\infty)$. By a result of Cesaratto and Vallee, this dimension spectrum tends to one exponentially fast, as α tends to infinity. As an analogue we investigate the cusp winding process for the geodesic flow on Schottky surfaces with cusps, and we show that the rate of convergence of the cusp winding spectrum is polynomial at infinity. This is a joint work with Yuya Arima (Nagoya University).

Toghrul Karimov
Max Planck Institute for Software Systems

Title From word combinatorics to decidability of logical theories

Abstract Recently, a string of results in computer science have emerged that show decidability or undecidability of certain classical logical theories using, among other concepts, toral translations, normality, numeration systems, and factor complexity. Briefly, the decision problem of a mathematical structure M over a language L is to decide, given a sentence $s \in L$, whether s is true in M. Below are a few examples which I will discuss. For a positive integer x, write $\mathrm{Pow}(x) = \{x^n : n \geq 0\}$. (Result 1) P. Hieronymi and C. Schulz have shown that the first-order theory of is undecidable. That is, no algorithm exists that verifies the truth of an arbitrary statement constructed from variables ranging over the naturals, predicates " is a power of 2 " and " is a power of 3 ", and symbols $0, 1, 2, \ldots, +, <$, " forall ", " exists ", " and ", " or ", " not ". The main tool is a certain novel normality property of binary expansions of powers of 3. (Result 2) In a recent joint work with V. Berthé et al. we show that (i) for any $a_1, \ldots, a_k > 0$ the monadic-second order (MSO) theory of is decidable, and (ii) the MSO theory of $= 0\} >$ is decidable assuming the square root of 2 is weakly normal in base 2. As a first approximation, MSO expands first-order logic with the ability to count the number of objects satisfying any (e.g., first-order) property modulo some number. The result (i) above is intimately connected to combinatorial properties of hypercubic billiard words.

Kiko Kawamura
University of North Texas

Title The Higher order partial derivatives of Okamoto's functions with respect to the parameter

Abstract In 2005, Okamoto introduced a one-parameter family of self-affine functions $\{F_a; 0 < a < 1\}$ defined on the interval $[0, 1]$. Although special cases of this family had been considered before (e.g. $F_{5/6}$ is Perkins' function, $F_{1/2}$ is Cantor's devil staircase and $F_{2/3}$ was considered independently by Bourbaki and Katsuura), Okamoto was the first to systematically study the differentiability of F_a for all $a \in (0, 1)$. It is well known that $F_a(x)$ is real analytic in a for every $x \in [0, 1]$. We introduce the functions

$$M_{k,a}(x) := \frac{\partial^k F_a(x)}{\partial a^k}$$

We will compute the box-counting dimension of the graph of $M_{k,a}$, characterize its differentiability, and investigate in detail the set of points where $M_{k,a}$ has an infinite derivative.While some of our results are similar to the known facts about Okamoto's function, there are also some notable differences and surprising new phenomena that arise when considering the higher order partial derivatives of F_a. This is joint work with Pieter Allaart, Nathan Dalaklis, Matthew Ortiz and Jiajie Zheng.

Makoto Kawashima
Meijigakuin University

Title On the linear independence of p-adic polygamma values

Abstract This is a joint work with A. Poëls (Lyon 1). In this talk, I shall introduce a new linear independence criterion for values of the p-adic polygamma functions defined by J. Diamond. As an application, we obtain the linear independence of some families of values of the p-adic Hurwitz zeta function $\zeta_p(s,x)$ at distinct shifts x. This improves and extends a previous result due to P. Bel, as well as irrationality results established by F. Beukers. Our proof is based on a novel and explicit construction of Padé-type approximants of the second kind of Diamond's p-adic polygamma functions. This construction is established by using a difference analogue of the Rodrigues formula for orthogonal polynomials.

Derong Kong
Chongqing University

Title Open dynamics with a moving hole

Abstract Motivated by the study of badly approximated numbers in Diophantine approximations, we consider the expanding map $T_b\colon x \mapsto bx \pmod 1$ on the unit circle with a moving hole. More precisely, given a sequence of open balls $\{B_n\}$ we study the set $K(\{B_n\})$ of x such that $T_b^n(x)$ avoids the ball B_n for all but finitely many n. We show that $K(\{B_n\})$ has full Hausdorff dimension if and only if the size of B_n tends to 0 as n goes to infinity. Our study of $K(\{B_n\})$ is also related to the products of random matrices. Some multifractal results on $K(\{B_n\})$ are also presented. This is ongoing joint work with Beibei Sun and Zhiqiang Wang.

Savinien Kreczman
Université de Liège

Title The Maximal Digit Property for numeration systems

Abstract We present the Maximal Digit Property of alternate base numeration systems, and show that it is equivalent to several interesting properties of those systems: Equality between spectra and beta-integers, optimality of representations, confluence of an associated rewriting system, and the possibility of normalizing representations with this rewriting system. In doing so, we generalize to an alternate base and merge several notions that were studied independently for Rényi bases. Joint work with Èmilie Charlier, Zuzana Masáková and Edita Pelantová.

Dominik Kwietniak
Jagiellonian University in Kraków

Title Bowen's Problem 32 and complexity of the conjugacy problem for systems with specification

Abstract We show that Bowen's Problem 32 on the classification of symbolic systems with specification does not admit a solution that would use concrete invariants. To this end, we construct a class of symbolic systems with specification and show that the conjugacy relation on this class is too complicated to admit such a classification. More generally, we gauge the complexity of the classification problem for symbolic systems with specification. Along the way, we also provide answers to two question related to complexity of pointed systems with specification: to a question of Ding and Gu related to the complexity of pointed Cantor minimal systems and to a question of Bruin and Vejnar on the complexity of the classification of pointed transitive Hilbert cube systems. This is joint work with Konrad Deka, Bo Peng and Marcin Sabok, see arXiv:2501.02723 https://doi.org/10.48550/arXiv.2501.02723

Seul Bee Lee
Seoul National University

Title Moment formulas of Siegel transforms with congruence conditions in dimension 2

Abstract The Siegel transform was developed to study the average properties of lattice points in the real plane. It has been applied to problems in counting theorems and Diophantine approximation. In this talk, we introduce Siegel transforms associated with principal congruence subgroups, along with their first and second moment formulas. We then consider problems of counting primitive lattice points satisfying congruence conditions. We apply these results to obtain analogs of Schmidt's random counting theorem and a quantitative Khintchine theorem. This is joint work with Jiyoung Han.

Yao-Qiang Li
Guangdong University of Technology

Title Spectrality and supports of infinite convolutions in R^d

Abstract We study the spectrality of a class of infinite convolutions in R^d, generalizing a result given by Li, Miao and Wang in 2022 from R to R^d. This allows us to easily construct spectral measures with and without compact supports in R^d, and motivates us to systematically study the supports of infinite convolutions. In particular, we give a sufficient and necessary condition for infinite convolutions to exist with compact supports, generalizing a related well-known result which is widely used. After giving strong relations between supports of infinite convolutions and sets of infinite sums, we study the closedness and fractal dimensions of infinite sums of union sets in order to deal with non-compact supports of infinite convolutions. As an application of these new tools, we deduce that there are spectral measures with and without compact supports of arbitrary Hausdorff and packing dimensions in R^d, generalizing another result given by Li, Miao and Wang in 2022 from R to R^d.

Jun Luo
Chongqing University

Title Spectral properties of Moran-type measures with staggered contraction ratios

Abstract Consider a Moran-type iterated function system (IFS) $\{\phi_{k,d}\}_{d\in D_{2p_k},k\geq 1}$, where each contraction map is defined as

$$\phi_{k,d}(x) = (-1)^d b_k^{-1}(x+d),$$

with integer sequences $\{b_k\}_{k=1}^{\infty}$ and $\{p_k\}_{k=1}^{\infty}$ satisfying $b_k \geq 2p_k \geq 2$, and digit sets $D_{2p_k} = \{0, 1, \ldots, 2p_k - 1\}$ for all $k \geq 1$. We first prove that this IFS uniquely generates a Borel probability measure μ. Furthermore, under the divisibility constraints

$$p_2 \mid b_2, \quad 2 \mid b_2, \quad \text{and} \quad 2p_k \mid b_k \text{ for } k \geq 3,$$

with $\{b_k\}$ bounded, we demonstrate that μ constitutes a spectral measure, that is, $L^2(\mu)$ admits a family of orthonormal basis of exponentials. To fully characterize the spectral properties, we introduce a multi-stage decomposition strategy for spectrums. By imposing the additional hypothesis that all parameters p_k are even, we establish a complete classification of μ's spectrality. This result unifies and extends the frameworks proposed in [An-He2014, Deng2022, Wu2024], providing a generalized criterion for such measures.

Stefano Marmi
Scuola Normale Superiore

Title The Brjuno and Wilton Functions

Abstract The Brjuno and Wilton functions bear a striking resemblance, despite their very different origins; while the Brjuno function is a fundamental tool in one-dimensional holomorphic dynamics, the Wilton function stems from the study of divisor sums and self-correlation functions in analytic number theory. We show that these perspectives are unified by the semi-Brjuno function . Namely, and can be expressed in terms of the even and odd parts of , respectively, up to a bounded defect. Based on numerical observations, we further analyze the arising functions and , the first of which is Hölder continuous whereas the second exhibits discontinuities at rationals, behaving similarly to the classical popcorn function. Joint work with Claire Burrin and Seul Bee Lee. Preprint available at arXiv:2503.08206

Nathaniel Nollen
University of the Philippines Diliman

Title Rotational beta expansions and Schmidt games

Abstract We consider rotational beta expansions in dimensions 1, 2 and 4 and view them as expansions on real numbers, complex numbers, and quaternions, respectively. We give sufficient conditions on the parameters $\alpha, \beta \in (0, 1)$ so that particular cylinder sets arising from the expansions are winning or losing Schmidt (α, β)-game.

Jorge Olivares-Vinales
Fudan University

Title Cantor attractor of Fibonacci-like unimodal maps

Abstract The Fibonacci-like unimodal maps that have been studied in recent years give rise to zero-entropy minimal Cantor systems. In this talk we present an explicit construction of these Cantor systems in the real line, how to obtain Bratelli-Vershik systems from them, and some applications of these constructions. This is joint work with Semin Yoo.

Antoine Renard
Université de Liège

Title Automatic proofs in combinatorial game theory

Abstract In this talk, we consider Wythoff's game and many variations studied by Fraenkel and others. In this game, two players take turns removing tokens from two piles with the following rules: either you take some tokens from only one of the two piles, or you withdraw the same amount from both. The first player unable to play loses the game. More precisely, we are interested here in characterising the P-positions (i.e. the losing positions) of the game. We present Walnut, a software commonly used in combinatorics on words, and show how to use it to obtain short automatic proofs of several results from the literature, as well as a long-standing conjecture stated by Duchêne et al. regarding additional moves not changing the set of P-positions. Moreover, Walnut allows us to state new results and conjectures about generalisations of Wythoff's game. This work is linked with non-standard numeration systems for which addition is recognisable by a finite automaton. In particular, we make use of the works of Frougny and Sakarovitch to build an adder for these specific numeration systems. The talk is based on a joint work with Bastien Mignoty, Michel Rigo and Markus Whiteland.

Manon Stipulanti
Université de Liège

Title Computing generalized abelian complexities of a family of sequences in effective ways

Abstract In combinatorics on words, a conjecture due to Parreau et al. in 2015 claims that, given an abstract numeration system S, the k-abelian complexity (a generalization of the abelian complexity due to Karhumäki et al. in 2013) of an S-automatic sequence is itself S-regular. Some isolated instances for this conjecture have been identified, but no large family of sequences. In this talk, such a family of sequences is exhibited, which are fixed points of Pisot-type substitutions and for which the abstract numeration system is the classical Dumont-Thomas numeration system associated with the substitution. As a particular case, the famous Tribonacci sequence is examined and new properties of this sequence are highlighted. This is a joint work with J.-M. Couvreur (Orléans, France), M. Delacourt (Orléans, France), N. Ollinger (Orléans, France), P. Popoli (Liège, Belgium), and J. Shallit (Waterloo, Canada).

Hiroki Takahasi
Keio University

Title Fractal counterparts of density combinatorics theorems in continued fractions

Abstract We build a bridge from density combinatorics to dimension theory of continued fractions. We establish a fractal transference principle that transfers common properties of subsets of $\mathbb{N}$ with positive upper density to properties of subsets of irrationals in $(0,1)$ for which the set $\{a_n(x)\colon n\in\mathbb{N}\}$ of partial quotients induces an injection $n\in\mathbb{N}\mapsto a_n(x)\in\mathbb{N}$. Let $(*)$ be a certain property that holds for any subset of $\mathbb{N}$ with positive upper density. The principle asserts that for any subset S of $\mathbb{N}$ with positive upper density, there exists a set E_S of Hausdorff dimension $1/2$ such that the set $\bigcup_{n\in\mathbb{N}}\bigcap_{x\in E_S}\{a_n(x)\}\cap S$ has the same upper density as that of S, and thus inherits property $(*)$. Examples of $(*)$ include the existence of arithmetic progressions of arbitrary lengths and the existence of arbitrary polynomial progressions, known as Szemerédi's and Bergelson-Leibman's theorems respectively. In the same spirit, we establish a relativized version of the principle applicable to the primes, to the primes of the form y^2+z^2+1, to the sets given by the Piatetski-Shapiro sequences.

Lama Tarsissi
Sorbonne University Abu Dhabi

Title The distribution of minimal fractions on the Stern-Brocot tree

Abstract The Stern-Brocot (SB) tree enumerates all irreducible rationals, uniquely positioned with ties to continued fractions and Christoffel words. In this work, we study a quotiented SB tree, focusing on fractions at levels multiple of 3 and their presence on minimal paths defined by the second-order balancedness of Christoffel words. We describe the distribution of all fractions belonging to a minimal path at these levels, detailing their positions, and provide a comprehensive study of the gaps between consecutive fractions, revealing structural patterns tied to the quotiented tree. In prior work, fractions Fn Fn+1 with Fn from the Fibonacci sequence were found to be on a minimal path (zigzag path). Analogously, we study four other minimal paths that define the boundary of the set of all minimal paths.

Maria Clara Werneck
Université Paris Cité

Title Minimal Symbolic Discrepancy for Tribonacci System

Abstract R. Tijdeman provided an upper bound over frequencies for the infimum of discrepancy of words over a finite alphabet, which measures how far the occurrences of each letter can deviate from their frequencies. However, for a fixed shift symbolic dynamical system, it remains unclear whether the sequence achieving low discrepancy proposed by R. Tijdeman is in the system. In this talk, we will explicitly show such sequences for the Tribonacci system and entirely relate discrepancy values with words in the Fibonacci system. The presented ideas can be used to prove low discrepancy for k-bonacci systems.

Posters

Hiroaki Ito

Montanuniversität Leoben

Title Convergence and Combinatorics of the Reverse Algorithm

Abstract We studied the Reverse algorithm, which is the Multidimensional continued fraction. Then, we showed that the algorithm is ergodic and the second Lyapunov exponent is negative. Furthermore, we got some balanced properties of the words generated from their associated substitutions. This is a joint work with Niels Langeveld and Jörg Thuswaldner.

Satoru Oshima

University of Tsukuba

Title Linear independence of certain infinite series through the application of word combinatorics

Abstract P. Erdős (1957) proved that for any integer $b \geq 2$, if a sequence of positive integers $\{k_n\}_{n\geq 0}$ is strictly increasing and satisfies $\limsup k_n/n^\ell = \infty$, then $\sum_{n\geq 0} b^{-k_n}$ is either a transcendental number or an algebraic number of degree at least $\ell + 1$. We focused on numbers for which Erdős's theorem cannot be applied. In this talk, we consider S_d $(d = 0, 1, 2)$, the set of all non-negative binary integers where the digits at positions not congruent to $d \pmod 3$ are zero. The set S_d increases roughly at an order of N^3 as a sequence, but the difference between consecutive terms oscillates. We prove the linear independence of $\{1\} \cup \{(\sum_{s\in S_d} t_d(s) b^{-s})^3\}_{d=0,1,2}$. This result is derived using Minkowski sums and word combinatorics. Furthermore, the proof is flexible in the sense that it may handle languages other than S_d, and the condition on the sequence of coefficients $t_d(s) \in \mathbb{Z}_{\geq 1}$ is only an appropriate upper bound of exponential order.

Lucía Rossi

Vienna University of Technology

Title Algebraic Number Systems

Abstract Given an algebraic number alpha (in principle, we assume it is a Gaussian rational with modulus greater than one), we introduce an algorithm to find a digit expansion of a given complex number in this base, using a suitable set of digits of the form $\{0, 1, ..., a\}$, which differs from the greedy algorithm. This expansion generalizes rational base number systems introduced by Akiyama, Frougny and Sakarovitch and is related to rational self-affine tiles introduced by Steiner and Thuswaldner by considering certain p-adic completions in terms of Gaussian primes. We study the language of these expansions, the problem of convergence, and relate them to fractal tiles.

Jihyuk Seo
Sorbonne Université

Title Transcendence of numbers related to S-adic sequences

Abstract After the Hartmanis-Stearns conjecture, the transcendence of numbers with β-expansions in algebraic bases β related to morphic sequences has been studied. Recently, Pavol Kebis, Florian Luca, Joël Ouaknine, Andrew Scoones, and James Worrell proved that such expansions yield transcendental numbers when given by so-called echoing sequences. They also showed that Sturmian and, more generally, Arnoux-Rauzy sequences, are echoing. We extend these results to S-adic sequences such that the associated S-adic shift has purely discrete spectrum and the condition PRICE of Berthé, Steiner and Thuswaldner (2019) holds. Our approach builds upon the geometrical interpretation of these sequences through their Rauzy fractals. This is joint work with Wolfgang Steiner.

David Siukaev
Université Sorbonne-Paris-Nord, LAGA

Title Invariant measures for some multidimensional fractions

Abstract Multidimensional continued fraction algorithms are generalizations of the classical continued fraction algorithm, namely Euclid algorithm and Gauss algorithm as its projectivised version. We want to find invariant measures for particular MCF algorithms. To do this, we suggest a natural extension for an algorithm acting on a graph (which is called win-lose induction and is inspired by the notion of Rauzy-Veech induction for interval exchanges) and construct it using the formalism of simplicial systems developed by C. Fougeron. The action of win-lose induction on the graph under consideration should be dynamically conjugated to the initial MCF algorithm. We discuss applications to the question of convergence of the MCF algorithms.
Key words: Continued fractions, invariant density, win-lose induction, natural extension, simplicial systems

Beibei Sun
Chongqing university

Title Open dynamics with a moving hole

Abstract Given an integer $b \geq 3$, let $T_b : [0,1) \to [0,1); x \mapsto bx \pmod 1$ be the expanding map on the unit circle. For any $m \in \mathbb{N}$ and for any $\omega = \omega^0\omega^1 \ldots \in (\{0,1,\ldots,b-1\}^m)^{\mathbb{N}}$ let

$$K^\omega = \{x \in [0,1) : T_b^n(x) \notin I_{\omega^n} \ \forall n \geq 0\},$$

where I_{ω^n} is the b-adic basic interval generated by ω^n. Then K^ω is called the survivor set of the open dynamical system $([0,1), T_b, I_\omega)$ with respect to the sequence of holes $I_\omega = \{I_{\omega^n} : n \geq 0\}$. We show that the Hausdorff and lower box dimensions of K^ω always conincide, and the packing and upper box dimensions of K^ω also coincide. Moreover, we give sharp lower and upper bounds for the dimensions of K^ω, which can be calculated explicitly. Finally, for any admissible $\alpha \leq \beta$ we show that there exist infinitely many ω (in fact of positive dimension) such that $\dim_H K^\omega = \alpha$ and $\dim_P K^\omega = \beta$.

As applications we first study badly approximable numbers in Diophantine approximation. For an arbitrary sequence of balls $\{B_n\}$ on $[0,1)$, let $K(\{B_n\})$ be the set of $x \in [0,1)$ such that $T_b^n(x) \notin B_n$ for all but finitely many $n \geq 0$. Assuming the limit $\lim_{n\to\infty} \operatorname{diam}(B_n)$ exists, we show that $\dim_H K(\{B_n\}) = 1$ if and only if $\lim_{n\to\infty} \operatorname{diam}(B_n) = 0$. Similarly, our results can be applied to the set of non-recurrence points. For any positive function ϕ on $\mathbb{N}$, let $E(\phi)$ be the set of $x \in [0,1)$ satisfying $|T_b^n(x) - x| \geq \phi(n)$ for all but finitely many n. Then we prove, under the existence of the limit $\lim_{n\to\infty} \phi(n)$, that $\dim_H E(\phi) = 1$ if and only if $\lim_{n\to\infty} \phi(n) = 0$. Our results can also be applied to study joint spectral radius of matrices. We show that the finiteness property for the joint spectral radius of associated adjacency matrices holds true.

Takafumi Tsurumaki
University of Tsukuba

Title On sufficient conditions for the linear independence of infinite series with exponents n^k

Abstract For integers $b, k \geq 2$, define $\gamma_k^{(b)} := \sum_{n=0}^{\infty} b^{-n^k}$. By studying the average behavior of nonzero digits of algebraic numbers, P. Erdos (1957) proved that $\gamma_k^{(b)}$ is either a transcendental number or an algebraic number of degree at least k. Later, H. Bailey, M. Borwein, E. Crandall, and C. Pomerance (2004) established a lower bound for the average number of nonzero digits in the binary expansion of algebraic numbers. As a consequence, they provided another proof of Erdős's result in the case $b = 2$. Moreover, a result by Y. Tachiya and S. Murakami (2024) is known: in the special case, they showed that $1, \gamma_k^{(b)}$ $(k = 2, 3, \cdots)$ are linearly independent over $\mathbb{Q}$. However, the algebraic independence of $\gamma_k^{(b)}$ remains an open problem. In this talk, we present sufficient conditions for nonzero polynomials $P(X,Y) \in \mathbb{Z}[X,Y]$ to satisfy $P(\gamma_k^{(b)}, \gamma_\ell^{(b)}) \neq 0$. This result is based on the proof method of Kaneko's criterion for algebraic independence.

Kosaku Watanabe
Hiroshima University

Title The fractal structure in the P-positions of variations of Wythoff Nim

Abstract In the combinatorial game theory, the Wythoff Nim is a well-known variant of classical 2-heaps Nim : the player can take either 1 or more tokens from one heap, or take one or more same number of tokens from both heaps. We can describe P-position which second player can force win of this game to use the golden ratio. We consider generalizations of the classical two heaps Wythoff Nim: For the integers "a" and "c" where $1 \leq a$ and $0 \leq c$, in (a, c)-Wythoff Nim, the player can take either "a" or more tokens from one heap, or take tokens from both heaps at least 1 token such that the difference of removed tokens is less than or equal to "c". In this talk, we describe the set of P-positions of the normal play of those games. Interestingly, a fractal structure by symbolic substitution appears in this set. The key idea is that every non-negative number appears exactly once at P-position where $x \leq y$, and the difference sequence of the P-position coordinates is stable to some substitution. This is a joint work with Shun-ichi Kimura and Takahiro Yamashita.

Takayuki Watanabe
Chubu University

Title On the connectedness of the limit set of non-autonomous IFS

Abstract Fractals are ubiquitous in nature, and since Mandelbrot's seminal insight into their structure, there has been growing interest in them. While the topological properties of the limit set of IFS have been studied—notably in the pioneering work of Hata—many aspects remain poorly understood, especially in the non-autonomous setting. In this paper, we present a homological framework which captures a structure of the random limit set. We apply our novel abstract theory to the concrete analysis of so-called fractal square, offering new insights into the topology of fractals.

Shuqin Zhang
Fudan University

Title Frostman and Fourier characterisations of fractal dimensions

Abstract We examine Frostman-type characterisations and other extremal measure criteria for a range of fractal dimensions of sets. In particular we derive properties of the less familiar modified lower box dimension and upper correlation dimensions. We also express a number of fractal dimensions in terms of Fourier properties of measures.

Yuhan ZHANG
College of Mathematics and Statistics, Chongqing University

TITLE Phase transitions for unique codings of fat Sierpinski gasket

ABSTRACT Given an integer $M \geq 1$ and $\beta \in (1, M+1)$, let $S_\beta(M)$ be the fat Sierpinski gasket in $\mathbb{R}^2$ generated by the iterated function system $\{f_d(x) = \frac{x+d}{\beta} | \beta \in \Omega_M\}$, where $\Omega_M = \{(i,j) \in \mathbb{Z}^2_{\geq 0} | 0 \leq i+j \leq M\}$. Then each $x \in S_\beta(M)$ may represent as a series $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} =: \pi_\beta((d_i))$ for some sequence $(d_i) \in \Omega_M^{\mathbb{N}}$. The infinite sequence (d_i) is called a *coding* of x. Since $\beta < M+1$, a point in $S_\beta(M)$ may have multiple codings. Let $U_\beta(M)$ be the set of $x \in S_\beta(M)$ having a unique coding, that is

$$U_\beta(M) = \{x \in S_\beta(M) | \#\pi_\beta^{-1}(x) = 1\}.$$

When $M = 1$, Kong and Li [2020, Nonlinearity] described the two critical bases for the phase transitions of the intrinsic univoque set $\widetilde{U}_\beta(1)$, which is a subset of $U_\beta(1)$. In this paper we consider $M \geq 2$, and characterize the two critical bases $\beta_G(M)$ and $\beta_c(M)$ for the phase transitions of $U_\beta(M)$: (i) if $\beta \in (1, \beta_G(M)]$, then $U_\beta(M)$ is finite; (ii) if $\beta \in (\beta_G(M), \beta_c(M))$ then $U_\beta(M)$ is countably infinite; (iii) when $\beta = \beta_c(M)$ then $U_\beta(M)$ is uncountable and has zero Hausdorff dimension; (iv) when $\beta > \beta_c(M)$ then $U_\beta(M)$ has positive Hausdorff dimension. Our results also apply for the intrinsic univoque set $\widetilde{U}_\beta(M)$. Moreover, we show that the first critical base $\beta_G(M)$ is a perron number, while the second critical base $\beta_c(M)$ is a transcendental number.

Yuru ZOU
Shenzhen University

TITLE Topological and dimensional properties of some sets in double-base expansions

ABSTRACT Given two real numbers q_0, q_1 with $q_0, q_1 > 1$ satisfying $q_0 + q_1 \geq q_0 q_1$, we call a sequence (d_i) with $d_i \in \{0,1\}$ a (q_0, q_1)-*expansion* or a *double-base expansion* of a real number x if

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q_{d_1} q_{d_2} \cdots q_{d_i}}, \text{ where } d_i \in \{0,1\} \text{ for all } i \in \mathbb{N}.$$

In this talk, we will present topological and dimensional properties of some sets in double-base expansions.

Pitch Talk Speaker Order

September 8th (Monday) 14:45-15:15
1. Jihyuk SEO
2. Takafumi TSURUMAKI
3. Satoru OSHIMA
4. David SIUKAEV
5. Hiroaki ITO
6. Kosaku WATANABE

September 9th (Tuesday) 16:00-16:30
1. Yuru ZOU
2. Lucía ROSSI
3. Yuhan ZHANG
4. Beibei SUN
5. Shuqin ZHANG
6. Takayuki WATANABE